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Application of Sequential Auction Techniques to Nonlinear Targeting Assignment for Space-Delivered Entry Vehicles

By

Brian Allan Stiles

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

Master of Science

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Note: The Views expressed in this thesis are those of the author and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the U.S. Government

ABSTRACT

Application of Sequential Auction Techniques to Nonlinear Targeting Assignment for Space-Delivered Entry Vehicles

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In the future, the arsenal of the U.S. military will include Space-delivered weapons, released by reusable launch vehicles. Entry vehicles released from the launch platforms will be capable of guiding to target locations throughout the world. In order to adequately incorporate these weapons into military plans, theater commanders will require sophisticated planning algorithms to maximize the likelihood of destroying the most important targets. This thesis develops a target assignment algorithm which uses a sequence of linear auctions to optimize the assignment of entry vehicles to weighted targets. This result can be improved over time by use of a directed search method, which uses numerous sequences of linear auctions to improve on solutions by eliminating poor assignments. These methods are compared to greedy methods, showing improvement in the assignment solution.

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Thank you [redacted], my wonderful wife, for all the understanding, love, and grace you have given me during the past few months. I've always known I could count on you for anything. You are God's perfect provision for me, and it has shown. I love you and look forward to the rest of our lives together.

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ASSIGNMENT

Draper Laboratory Report Number T-1477.

In consideration for the research opportunity and permission to prepare my thesis by and at The Charles Stark Draper Laboratory, Inc., I hereby assign my copyright of the thesis to The Charles Stark Draper laboratory, Inc., Cambridge, Massachusetts.

Brian A. Stiles April 23, 2004

Table of Contents

List of Figures

List of Tables

1 Introduction

When developing plans for a military campaign, analysts must determine which assets should be used to destroy which targets. For general air, sea, and land based assets, targeting considerations include distance or time to target, available munitions, how heavily the target is defended, and the value of the target. With the inclusion of space-delivered weapons, military planners will have the capability to rapidly destroy targets anywhere in the world. However, this capability comes with a relatively large price tag. Therefore, although space-delivered weapons can be used as a supplement for air, sea, or land based assets, they will generally be used to destroy high value, heavily defended targets, or time sensitive targets that cannot be destroyed quickly using other assets. In the case of targets that must be destroyed, such as Weapons of Mass Destruction (WMD), multiple space-delivered weapons may be used to target the same location to increase the probability that the target is destroyed.

To model the target assignment problem, effects of cross targeting and orbital mechanics must be considered. These effects cause the assignment problem to become nonlinear; the total utility does not equal the sum of the utility of each assignment. Modeling this type of problem, target assignment for entry vehicles attempts to develop an assignment of entry vehicles to targets that results in the largest probability of destruction for the most important targets.

1.1 General Target Assignment Problems

Target assignment problems using land, sea, and air based military assets generally are posed as resource allocation problems. These problems include an assignment of a set of resources (weapons) to a set of tasks (targets). As such, these problems can be solved using linear programming techniques. A recent thesis discussed the use of linear programming techniques for the targeting cycle for air based assets [28]. However, linear programming techniques cannot be used to solve problems where the combined value of assignments (utilities) is nonlinear, such as the case when cross targeting occurs [24]. Research in the area of dynamic resource allocation, where the value of assignments are dynamic, considers situations where the total value of assignments is based on the combination of all resources to all tasks. Although a formulation for these types of problems has recently been introduced [20], there is not a set method to solve these problems.

1.2 Path Planning Problems

Because each assignment of a target to an Entry Vehicle can be made in order, the target assignment problem can be posed as a path through a field of nodes. Path planning problems have been efficiently solved using dynamic programming methods, which work well for problems when decisions are made in stages and tradeoffs must be made between present and future utilities of assignments. Dynamic programming, however, cannot be used for problems when the utility from a group of assignments cannot be linearly combined [7]. A solution method that allows non-additive path utilities is the A* node

search method. The A* technique uses a combination of the ability to estimate the cost of completing a path and the ability to backtrack to find better paths, to find the optimal trajectory [15]. Although the A* method can solve problems with all of the characteristics of the assignment problem, since it is not possible to develop an effective estimate for the cost of making an assignment it is not feasible to use A* to solve the problem.

1.3 Auction Solution Methods

Auction algorithms can be used to solve assignment problems [2], [3], [4], [5], [6]. When solving nonlinear assignment problems, combinatorial auctions are generally used, allowing bidders to place bids on multiple items during each round [16], [22]. However, combinatorial auctions are computationally difficult to solve, and may not be an applicable way to solve problems when bidders or items cannot be grouped into lots based on similarities [12]. An alternative method to solve nonlinear assignment problems using auction involves solving a sequence of auctions [8]. By making assignments in sequence, linear techniques can be used to solve the nonlinear assignment problem. The proposed solution uses a Linear Auction to determine assignments for each sequence. Linear Auctions can guarantee an optimal solution to a linear assignment problem, and can effectively be used as components of larger problems [12]. Compared to other linear programming techniques, linear auctions are competitive in their solution times when run on serial computers, with much faster solution times when run on parallel computers [1]. Two variations are also proposed to improve on this solution. The first method attempts to estimate poor assignments, eliminating these assignments and recomputing a solution

in an attempt to improve on the solution. The second method perturbs the weights associated with targets in an attempt to reach other solutions near the optimal solution.

1.4 Thesis Overview and Content

The second chapter of this thesis begins with a background discussion on Space Based Entry Vehicles. This discussion includes a small background on the different vehicles, an explanation on how inclination and release time affect the locations each entry vehicle can target, and how cross targeting, inclination, and release time affect the probability a target will be destroyed by Entry Vehicles.

The third chapter includes the problem statement, assumptions, problem formulation, and a series of network flow diagrams to display how the nonlinear effects of cross targeting affect the size and complexity of the problem. This section explains how the problem can be viewed as a path planning problem, with discussion on the components that differentiate it from regular path planning problems.

The fourth chapter includes a discussion on different ways to incrementally solve the assignment problem as well as a methodology to determine the utility of individual assignments for use in the linear assignment methods. A description and background on the use of auctions in solving linear assignment problems is given. A comparison between the results of a linear auction and a greedy method are used to show the effectiveness of linear auctions, with a round by round example of how a linear auction reaches a solution. Convergence of auctions is discussed as well as the effectiveness of

using bid increments. The discussion on Linear Auctions is followed by an implementation of Linear Auctions in a directed search method, which allows an improvement in solutions by identifying 'poor' assignments and eliminating these assignments with the goal of reaching an improved solution. Discussion also includes the use of randomly perturbed weights in a linear auction to reach other solutions in an attempt to improve on existing solutions. A brute force and a random assignment method are also discussed for the purpose of gaining a baseline to compare results.

The fifth chapter displays the results of the different assignment methods to various assignment problems. Some of the problems include those expected to be seen by Entry Vehicles, as well as two problems that introduce deficiencies of different assignment methods. Throughout the different problems the effectiveness of the use of Linear Auctions in the assignment problem is discussed, as well as improvements from both the directed search and random weighted methods.

The sixth chapter gives an overview of the topics covered in this thesis, as well as a listing of possible future work in the area of target assignment for Space Based Entry Vehicles.

2 Background

In order to understand the methodology used to solve the target assignment problem it is essential to understand the basic capabilities of the Launch and Entry vehicles and the affect this has on their assignments. This section includes information on both the Launch Vehicle and Entry Vehicle, the use of Footprints to estimate assignable targets, and the combination used to determine a probability of success for destroying a target.

2.1 Launch and Entry Vehicle

The research and results in this thesis are based on estimated data about Space Operations Vehicles (SOVs) and Common Aero Vehicles (CAVs) performance. However, the methodology involved in the solutions and problems relates to any target assignment problems involved space delivered entry vehicles. For this reason, throughout this thesis the SOV will be referred to as a Launch Vehicle (LV) while a CAV will be referred to as an Entry Vehicle (EV). More specifics about SOVs and CAVs are given in Sections 2.1.1 through 2.1.3, with an example launch profile for a LV/EV described in Section 2.1.4.

2.1.1 Space Operations Vehicle

The Space Operations Vehicle (SOV) is a next generation reusable launch vehicle (LV) currently under development. It is currently projected as a two-stage to orbit launch vehicle, with the capability to fly sub-orbital "pop-up" trajectories to launch payload into

orbit. Its estimated capability includes orbiting 6,000 pounds or carrying 40,000 pounds of weapons through space on a once around orbit and returning to its launch site (the LV can carry more payload for a once around due to lower fuel requirements for lower orbits). As an unmanned vehicle, the first phase of the SOV will help boost the vehicle into orbit, and then fly back to its initial launch location. The second phase, which is projected to be unmanned, will provide the boost to obtain orbit or the desired sub-orbital "pop-up" while carrying the required payload [25].

2.1.2 Common Aero Vehicle

Common Aero Vehicles (CAVs) are small, unpowered, maneuverable entry vehicles (EV) capable of carrying various payloads. The CAV is essential a shell, with current designs giving downrange but little cross range maneuverability, with future lifting body designs capable of carrying more payload with better maneuverability. Planned payloads include a unitary penetrator which will use the CAV's high impact velocity (kinetic energy) as its kill mechanism rather than explosives, any current and future ground attack munitions, Unmanned Aerial Vehicles (UAVs), or Agent Defeat weapons for neutralizing biological or chemical weapons. Working in conjunction with SOVs as part of a future space-based global strike ability, CAVs would allow the United States the capability to strike anywhere in the world with a conventional attack within 6 hours of an order [25].

2.1.3 Operational Use

Due to the changing nature of the battlefield, LVs and EVs will be an integral part of future war fighting, both for their capability to strike targets quickly and their ability to supplement the aging bomber force. LVs will generally be used to eliminate high value targets, where their location is based on time sensitive information. In extended use campaigns, they can be launched as frequently as twice a day, with a twelve hour turn around time. As the battlefield changes, due to updated battlefield damage assessment, the LVs will be flexible to meet targeting needs.

An ideal situation for LV/EV use would be a first strike attack on an enemy with a small number of Weapons of Mass Destruction (WMD). In such a situation, other conventional attacks could be too slow, alerting the target nation and allowing them to use or effectively protect their WMDs. Assuming the lack of sophisticated space-based launch-sensing technology, LV/EVs could be used to search out and destroy the WMDs before the target nation could act. This would allow the United States the flexibility to attack a rogue nation which possesses WMDs, decreasing the probability that these weapons would be used. A similar situation would involve the necessity for an immediate counter-strike following an initial attack against the United States, or the need to destroy a remote target quickly based on time-sensitive intelligence [23].

2.1.4 Example Profile

An example LV/EV profile is shown in Figure 2-1. The LV/EV will be launched into a desired inclination, which is the angle between the LV/EVs orbital plane and the equator (See Section 2.2.2). During the boost phase the first stage will detach and return to the launch site. Before the Launch Vehicle enters a Low Earth Orbit (LEO), the Entry Vehicle will be released at its desired release time. Once released, the Entry Vehicle will be capable of reaching a certain space of land, designated in Figure 2-1 by a circle around the target. The Entry vehicle will follow a path to its desired target. The Launch Vehicle will continue into orbit, and will later deorbit to land back at its launch site.

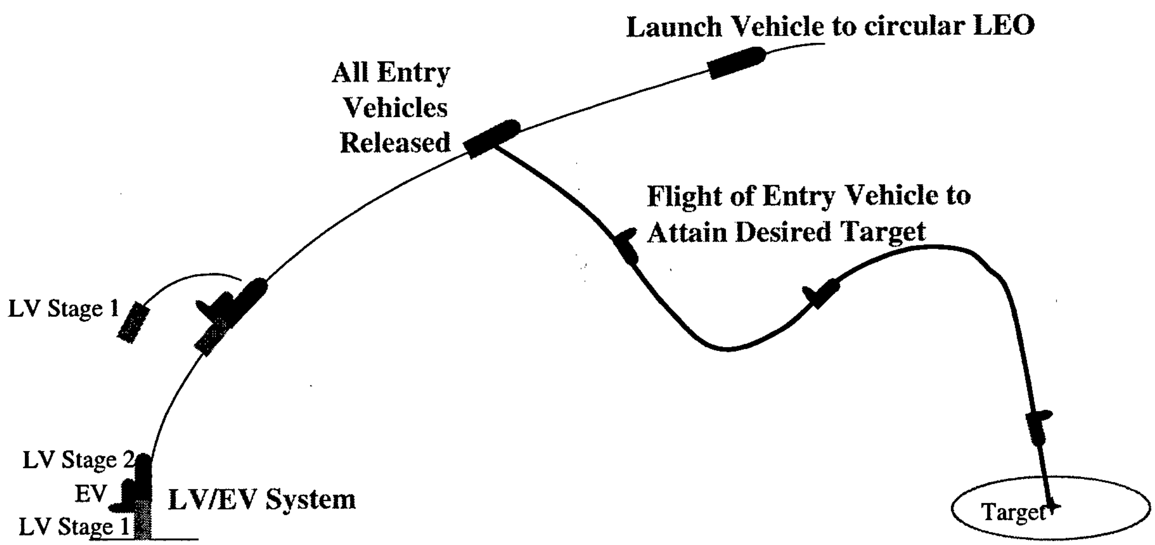


Figure 2-1. Example LV/EV Trajectory

2.2 Footprint

Once an Entry Vehicle is released from a Launch Vehicle, it is able to control its angle of attack (alpha) and bank (beta) angle allowing it to guide to a target. The *footprint* of the

vehicle is the total area in which an Entry Vehicle can hit a target. The borders of the footprint show the maximum region that the Entry Vehicle can target. Figure 2-2 gives an example of a footprint. The black dashed lines in the picture represent different trajectories leading to the edge of the footprint. Generally, the maximum downrange/cross range (top, bottom, and right edges of footprint) boundaries of the

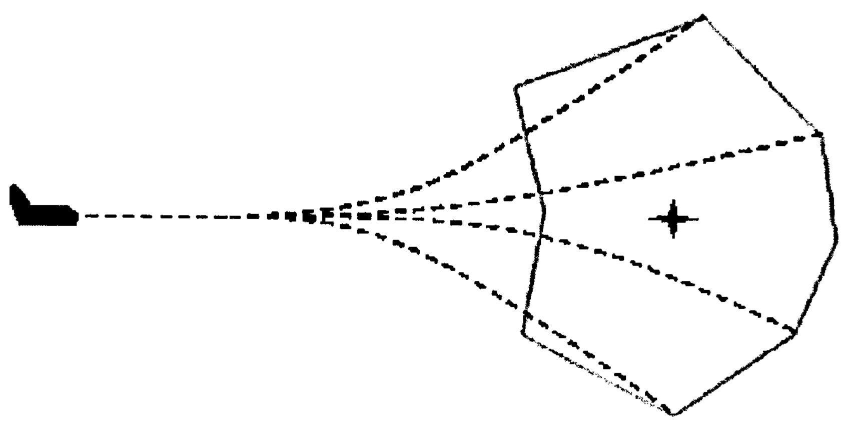


Figure 2-2. Entry Vehicle footprint with Flight Paths and Nominal Reference Center

footprint are limited by a vehicle's glide capability at the angle of attack that corresponds to the maximum lift to drag ratio. The minimum range (left edge of footprint) boundary is generally limited by vehicle constraints such as aerodynamic loading or thermal limits [14]. The red star in the center is the nominal profile reference center. This is the location that the Entry Vehicle would hit assuming flight along a nominal alpha and beta trajectory.

2.2.1 Effect of Release Time on Footprint

The relative size of a footprint is based on the energy of the vehicle. For Entry Vehicles, this energy is a combination of their release velocity and altitude. Since Entry Vehicles will gain both altitude and velocity at later release times, the higher the release time the larger the footprint. Similarly, since a Launch Vehicle is traveling around the earth, the location of release for each release time will be different. Because of this, later release times will results in footprints occurring farther downrange. Figure 2-3 shows an example of two different release points for an Entry Vehicle. The first footprint is both smaller in size and less downrange, while the second release time has a larger footprint and occurs farther downrange.

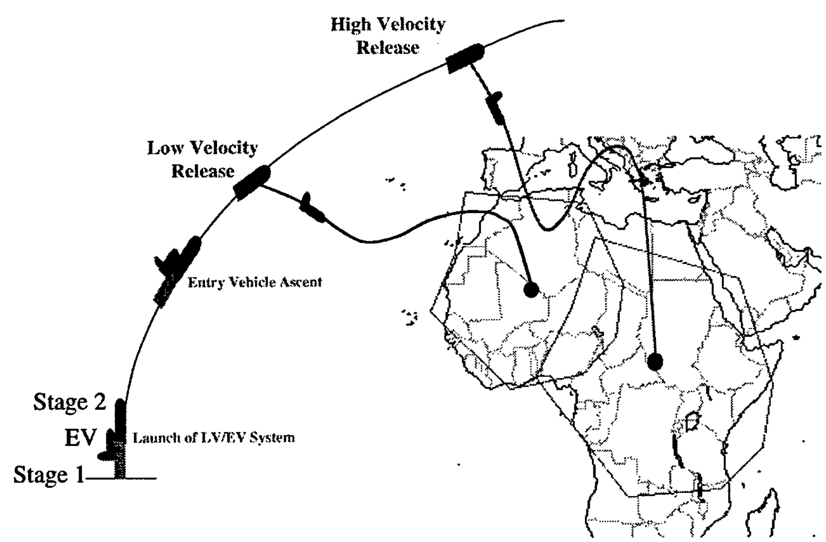
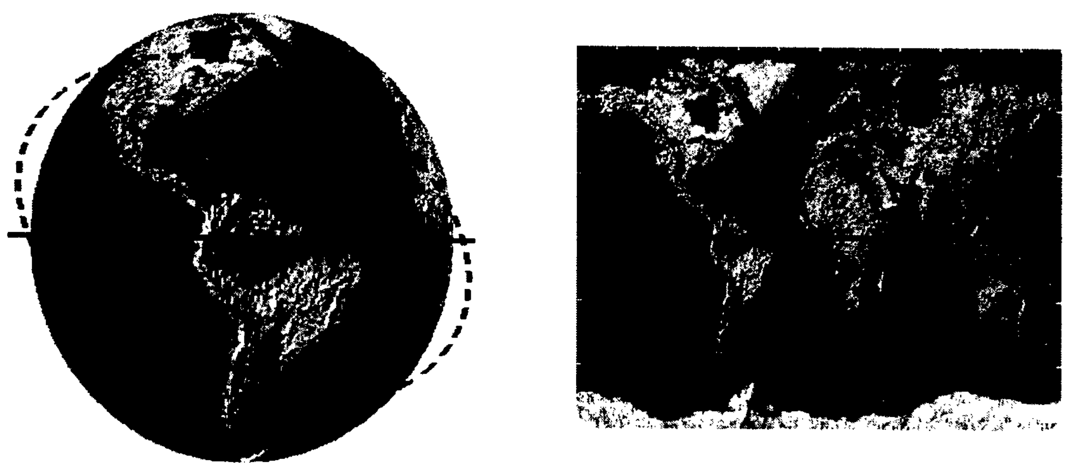


Figure 2-3. Variation in Footprints based on Release Time for East Coast US Launch

2.2.2 Effect of Inclination on Footprint

When a space vehicle enters into orbit, the path that it takes around the earth is guided by orbital mechanics. One of the parameters used to describe the orbit is "inclination." This is the angle between the orbit plane and the equator. An inclination of 0° will orbit about the equator, while 90° will orbit about both the north and south poles. Figure 2-4 displays two orbital paths (a path through an orbital plane) with different inclinations; the solid blue line is for a launch inclination of 0°, while the dashed red line is a launch inclination of 45°. The angle between the two paths is 45°, the difference in the inclination between the two paths. Figure 2-4a displays the paths around a spherical earth, while Figure 2-4b displays the paths using a cylindrical projection map. Because



(a) (b)

Figure 2-4. Inclination Diagram

each orbital path travels over different areas, Launch Vehicles traveling along different inclinations will have different footprints. An example of this is shown in Figure 2-5, where three foot prints from orbits with different inclinations, but the same release time, are shown. The three launch trajectories that correlate to the three footprints are shown with matching line color/style coming out of Florida.

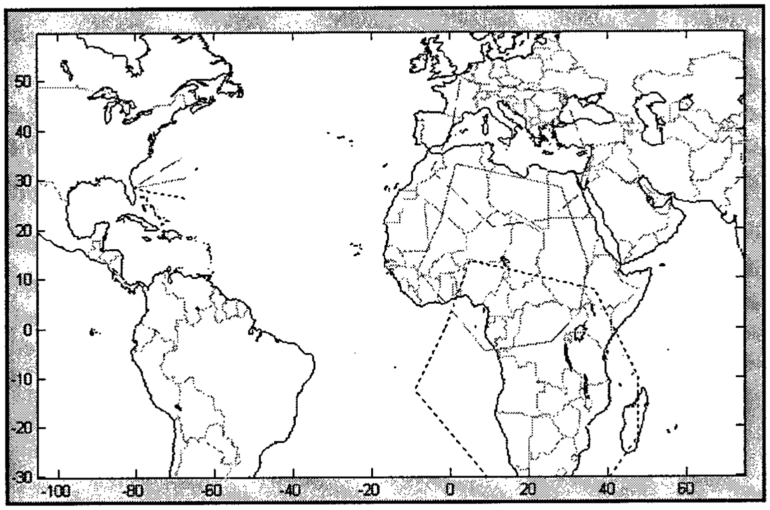


Figure 2-5. Variation in Footprint based on Launch Inclination

2.2.2.1 Region Covered by each Inclination

Theoretically each inclination has an infinite number of release times, and hence an infinite number of footprints to consider. Combining the region that all of these footprints cover gives the total region that a Launch Vehicle's Entry Vehicles may target. An example of one of these regions is shown in Figure 2-6. Here the outer edge of all footprints is shown with a thick red line indicating the swath of land that is targetable by

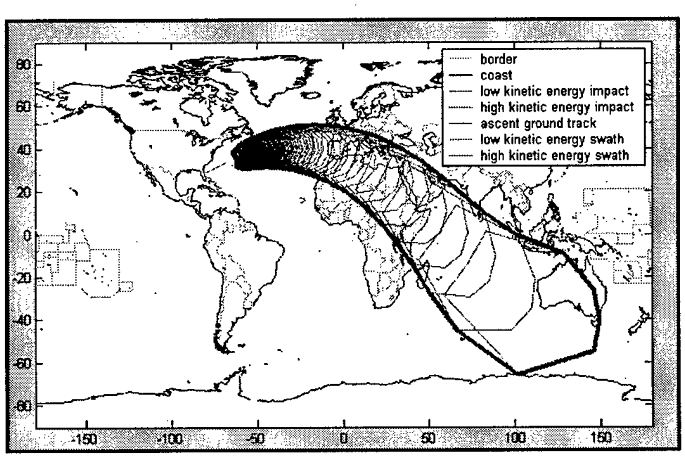


Figure 2-6. Footprint Swath with Individual Footprints

a Launch Vehicle's Entry Vehicle. Inside the swath are thinner red lines indicating the location of individual footprints. The image in Figure 2-7 shows swaths from the three launch inclinations in Figure 2-5. Areas that are covered by more than one swath can be

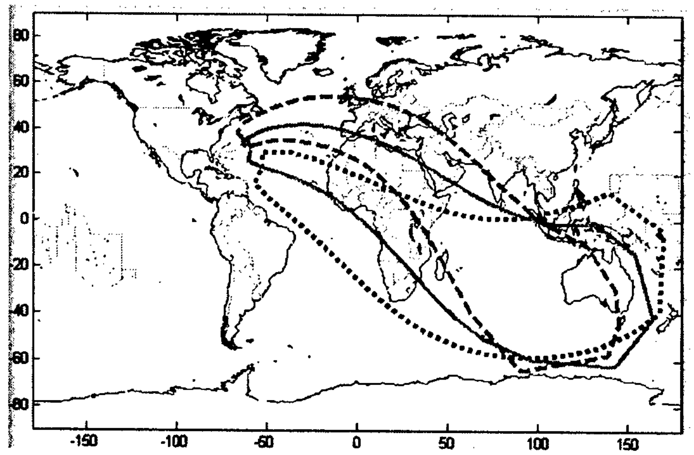


Figure 2-7. Swaths from 3 Launch Inclinations in Figure 2-5

targeted by multiple inclinations and multiple release times. An area in the Indian Ocean is covered by all three swaths because the orbital planes of the three inclinations cross over this area. Figure 2-8 shows a picture of all swaths used during simulation. These

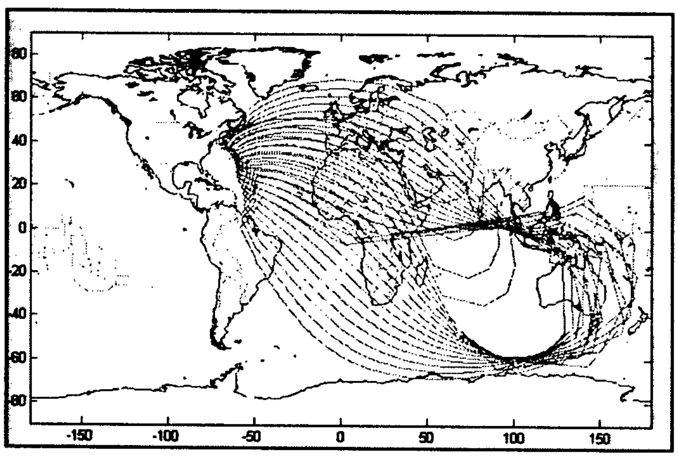


Figure 2-8. Swaths from numerous Inclinations showing Overlap

have numerous instances of overlap, leading to numerous inclinations and release times having footprints which cover the same area. Because there are more swaths overlapping

Africa than any other land mass, problems discussed in this thesis will involve targets within the continent of Africa.

2.3 Effect of Terminal Velocity on Footprint

The size and shape of a footprint is affected by the necessary terminal velocity at impact. Generally, footprints are calculated based on the maximum or minimum distance an Entry Vehicle can travel in a given direction. However, when an Entry Vehicle is used as a Unitary Penetrator, a different path to a target must be calculated to allow impact with a required terminal velocity (energy). Doing so changes the size and location of the footprint. Figure 2-9 displays two footprints for the same release time and inclination,

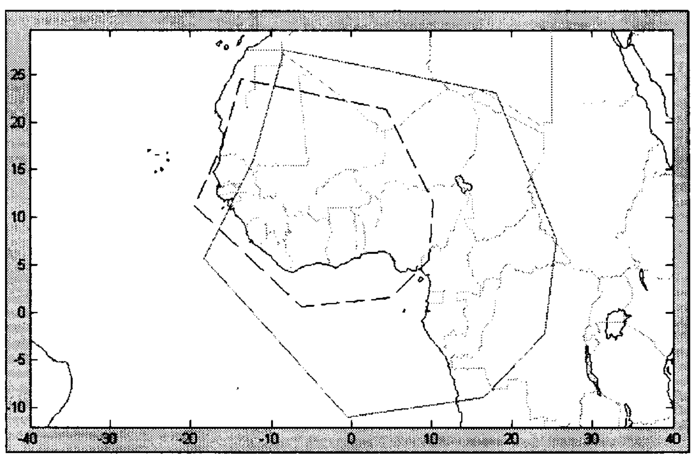


Figure 2-9. Footprint for Conventional and Unitary Penetrator Entry Vehicle

with the larger solid red footprint for a conventional Entry Vehicle, and the smaller dashed black footprint for a unitary penetrator Entry Vehicle. The unitary penetrator footprint occurs at a shorter downrange distance than the conventional footprint due to assumed different vehicle characteristics which allow the Entry Vehicle to fly a path to the target with an impact of desired velocity.

2.4 Probability of Success

The act of targeting does not guarantee a target will be destroyed. In order for a target to be destroyed, three events must occur: a Launch Vehicle must succeed in releasing an Entry Vehicle, an Entry Vehicle succeeds in flying to a target, and the release time and inclination chosen for the Entry Vehicle must provide adequate coverage for the Entry Vehicle to reach the target. Because of this, probability of successfully hitting a target is dependant on the inherent probability of failure of the Launch or Entry vehicles and the location of the target within the footprint.

2.4.1 Probability of Success from Cross targeting with multiple Entry Vehicles from one Launch Vehicle

The probability of success for destroying a target is related to what combination of Launch Vehicles and Entry Vehicles target it. Each vehicle has an inherent probability of failure (P_f). In order to determine the probability that a vehicle will succeed, the difference is taken between one and the P_f, resulting in a probability of success; the probability of success for a LV is denoted as $P(S_{LV})$. Because EVs are carried by launch vehicles, their probability of success is conditional, based on the assumption that the Launch Vehicle will succeed; this is shown as $P(S_{EV} \mid S_{LV})$ [13]. The combined probability of success for both a Launch Vehicle and entry vehicle is represented by

$P(S_{EV} \cap S_{LV})$. For a one Launch Vehicle, one Entry Vehicle targeting situation, if $P(S_{LV})$ is 0.8, and $P(S_{EV} \mid S_{LV})$ is also 0.8, then $P(S_{EV} \cap S_{LV})$ is 0.64 (equation 1).

$$P(S_{EV} \cap S_{LV}) = P(S_{LV})P(S_{EV} \mid S_{LV}) = (.8)(.8) = .64 \tag{1}$$

This combination is determined by multiplying the probability of success of the Launch Vehicle to the conditional probability of success of the Entry Vehicle. However, when the same target is targeted by multiple (2) Entry Vehicles from the same Launch Vehicle, the $P(S_{EV} \cap S_{LV})$ increases to 0.768 (equation 2) because the target has the chance to be destroyed by more than one entry vehicle.

$$\begin{aligned} P(S_{EV_{12}} \cap S_{LV}) &= P(S_{LV})(1-(1-P(S_{EV_1} \mid S_{LV}))(1-P(S_{EV_2} \mid S_{LV}))) \\ &= (.8)(1-(1-.8)(1-.8)) = .768 \end{aligned} \tag{2}$$

This calculation is more complicated because the probability of success of the LV remains the same, but the total conditional probability of success of the EVs becomes a combination of both EVs' conditional probability of failure (one minus $P(S_{EV} \mid S_{LV})$). Although targeting with an additional EV increases the $P(S_{EV} \cap S_{LV})$, the increase is limited due to the fact that both Entry Vehicles are carried by the same Launch Vehicle, hence limited in their total $P(S_{EV} \cap S_{LV})$ by the probability of success of the Launch Vehicle. In order to increase this limit, targets must be targeted by Entry Vehicles from different Launch Vehicles.

2.4.2 Probability of Success from Cross targeting with multiple Entry Vehicles from multiple Launch Vehicles

As stated in Section 2.4.1, the $P(S_{EV} \cap S_{LV})$ for a target, assuming it is targeted by two Entry Vehicles from the same Launch Vehicle, is 0.768. The value of $P(S_{EV} \cap S_{LV})$ from Entry Vehicles from a single Launch Vehicle is limited by the probability of success of the Launch Vehicle. However, in a situation where a target is again targeted by 2 Entry Vehicles, but this time each Entry Vehicle comes from a different Launch Vehicle, the $P(S_{EV} \cap S_{LV})$ increases to 0.87 (equation 3). Although this value is still limited by

$$\begin{aligned} P(S_{EV} \cap S_{LV_{12}}) &= 1-(1-P(S_{EV} \cap S_{LV_1}))(1-P(S_{EV} \cap S_{LV_2})) \\ &= 1-(1-.64)(1-.64) = .8704 \end{aligned} \tag{3}$$

the probability of success of the Launch Vehicles, because the limit on both is 0.8, their combined limit is 0.96; a substantial increase. Table 2-1 shows similar combinations. In

Table 2-1. Probability of Success from Cross Targeting

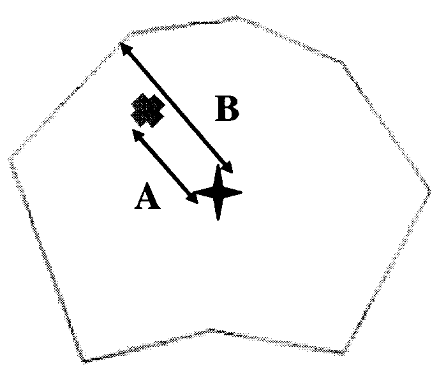
Total Number of EVs to Hit Target			
1	2	3	4
A1	A2	A3	A4
0.640	0.768	0.794	0.799
	A1 B1	A2 B1	A3 B1
	0.870	0.916	0.926
		A1 B1 C1	A2 B2
		0.953	0.946
			A2 B1 C1
			0.970
			A1 B1 C1 D1
			0.983

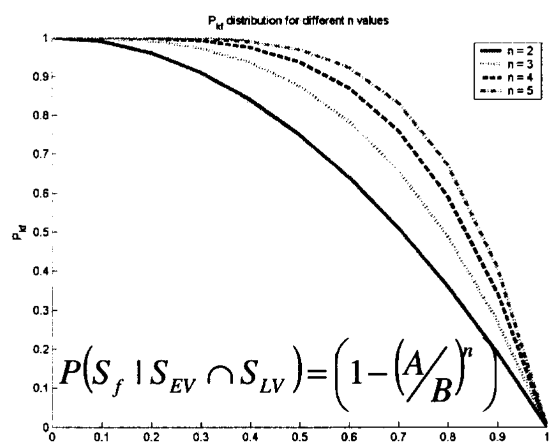
this chart, each column represents a number of Entry Vehicles targeting a target. The letters and numbers in the grey represent the number of Entry Vehicles from each Launch Vehicle (A through D) to target the target. The number in white below is the $P(S_{EV} \cap S_{LV})$. This diagram shows how important cross targeting is in order to

improve $P(S_{EV} \cap S_{LV})$. If targeting the same target with multiple EVs from a single LV, the maximum attainable $P(S_{EV} \cap S_{LV})$ is 0.799. However, it is possible to decrease the total number of EVs from four to two, and by cross targeting increase total $P(S_{EV} \cap S_{LV})$ to 0.87. For the fourth EV column, there is an increase of 0.184 (18.4%) when cross targeting from one Launch Vehicle to four Launch Vehicles.

2.4.3 Probability of Success from Target Location within a Footprint

Footprints are calculated based on mean environmental conditions. Actual environmental conditions can cause footprints to either change size or shape. High winds can cause footprints to shift in location as well as decrease in size, low density can cause larger footprints located further downrange; temperature and air pressure can also have an affect on footprint size or shape. In order to take this into account, a portion of the P_S comes





(a) (b)

Figure 2-10. Graphical Probability of Success from Target Location within Footprint

from the location of a target within a footprint. This is shown in Figure 2-10a, where the target is the magenta cross and the reference center is the red star. This is accomplished by comparing the radial distance between the reference center of the footprint and the target (A), and the reference center of the footprint and the outer edge of the footprint (B). Because the probability of success from target location within a footprint is conditioned on both the Launch Vehicle and Entry Vehicle succeeding, the probability is a conditional probability represented by $P(S_f \mid S_{EV} \cap S_{LV})$. As Figure 2-10b shows, this is based on a ratio of A and B. $P(S_f \mid S_{EV} \cap S_{LV})$ is one at the center, and approaches

$$P(S_f \mid S_{EV} \cap S_{LV}) = \left(1 - \left(A/B\right)^2\right) \quad (4)$$

zero as it reaches the outer edge. The amount of this decrease is based on the exponent. For the purpose of this research, a power of 2 was used. In order to more accurately represent the environmental effect on $P(S_f \mid S_{EV} \cap S_{LV})$ it would require a Monte Carlo analysis to determine a better evaluation of $P(S_f \mid S_{EV} \cap S_{LV})$. Figure 2-10b shows the affect of the power of $P(S_f \mid S_{EV} \cap S_{LV})$ on its values, with the current power of two represented by a solid blue line.

2.4.3.1 Discretization of Inclinations and Release Times

In order to pose the problem as an assignment problem, footprint inclinations and release times are discretized to limit the total number of possible footprints, hence treating probability of success from footprints as a table look up value rather than a value to be optimized based on system dynamics. For the purpose of this research, inclinations are

discretized in 5° increments. Figure 2-11 shows numerous footprints from one inclination. The upper left image includes a discretization with a release time Δt of 1

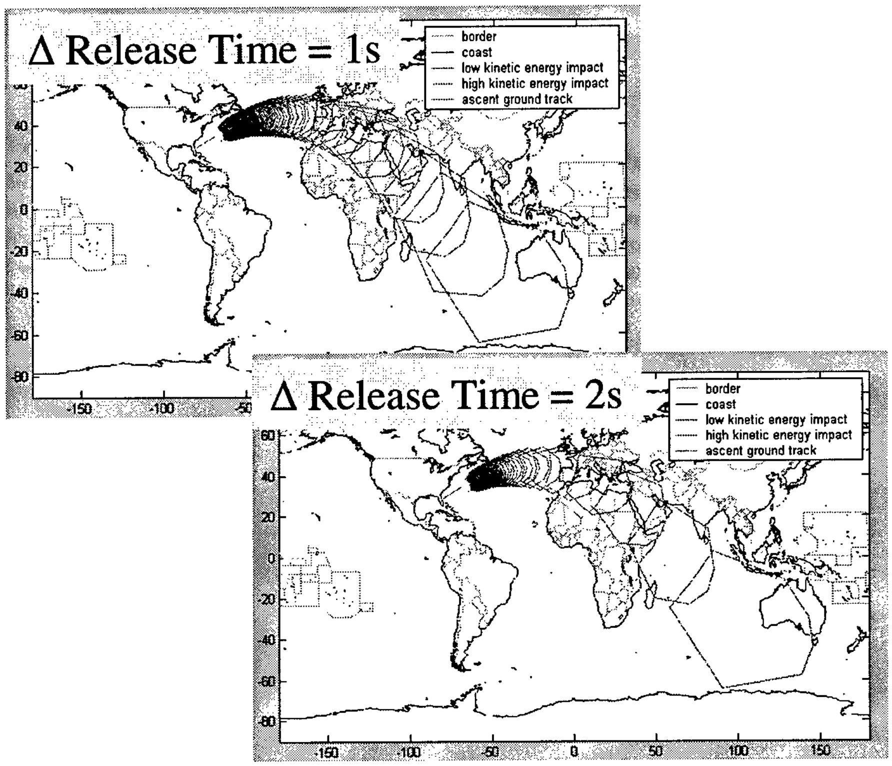


Figure 2-11. Discretized Footprints with varying Δ Release times

second, while the lower right image is for a Δt of 2 seconds. This discretization has decreased the number of possible $P(S_f \mid S_{EV} \cap S_{LV})$ values from an infinite number to a finite number. Discretization allows probability of success from target location within a footprints to be treated as a table lookup value, such as Table 2-2. This table is for a four target problem, with four release times for one launch inclination (t1, t2, t3, and t4). The values within each field are the $P(S_f \mid S_{EV} \cap S_{LV})$ for each target based on the location of targets within each footprint for each release time and inclination. Because of the numerous release times and inclinations, an actual chart of all values is considerably larger, with a $P(S_f \mid S_{EV} \cap S_{LV})$ chart for each inclination. This chart is developed a

Table 2-2. $P(S_f|S_{EV} \cap S_{LV})$ for Discretized Release times for one Inclination

$P(S_f \mid S_{EV} \cap S_{LV})$ values for one Inclination					
		Release Time			
		t1	t2	t3	t4
Target	1	0	0.1	0	0
	2	0	0.2	0.3	0.3
	3	0	0	0.25	0.7
	4	0.25	0.5	0.25	0

priori, and is used when calculating utilities for assignment; this is discussed later in Section 4.2.

2.4.4 Total Probability of Success

In order to determine a total P_S, both probabilities from vehicle failure and footprints must be combined. This leads to the simple equation (equation 5) where total probability

$$P_S = P(S_{EV} \cap S_{LV})P(S_f \mid S_{EV} \cap S_{LV}) \tag{5}$$

of success equals a multiplication between both subsets of probability of success. This simple equation will be used to determine total probability of success, while the next section will discuss the objective function, which includes a combination of probability of success and target weighting.

3 Problem Statement

Given a prioritized target list (weights assigned to each target), a warhead requirement (whether a target should be destroyed with a conventional EV or a unitary penetrator), and a number of Launch Vehicles and Entry Vehicles with associated mechanical probabilities of success, assign Entry Vehicles to targets and determine Launch Vehicle inclinations and release times in order to maximize the probability of success of hitting the highest weighted targets.

3.1 Assumptions

Since many of the specifics of both Launch Vehicles and Entry Vehicles are still unknown, a number of assumptions were used during the course of this problem:

I. Each Launch Vehicle can carry up to four Entry Vehicles

Although it is not currently known how many Entry Vehicles each Launch Vehicle can carry, using an estimate of four allows for cross targeting within EVs from the same LV, but doesn't lead to a situation with so many EVs that target assignment becomes trivial. This is shown as a constraint in equation 7.

II. Each Entry Vehicle may only be assigned to one target

Because an entry vehicle is destroyed when impacting a target, each Entry Vehicle will only be able to be assigned to one target. Because of this, each Launch Vehicle may target a maximum of four targets; one for each Entry Vehicle. This is represented by the constraint in equation 8.

III. Each Launch Vehicle must release all of its Entry Vehicles simultaneously

This assumption is based on the assumption that a Launch Vehicle will be forced to shutdown engines during payload release, a process that only occur once during each flight. This leads to each Launch Vehicle's Entry Vehicles having the same footprint, limiting these Entry Vehicles' assignment to targets within the same footprint. This acts both to simplify the problem by forcing Entry Vehicles to target the same area, and complicate the problem by decreasing the region that each Launch Vehicle can target.

IV. Entry Vehicles must be released before a Launch Vehicles reaches orbital velocity

Entry Vehicles will not contain boosters; hence have no ability to change their velocity in space. This decreases the number of release times for each inclination.

V. Probability of Success for LVs, and conditional Probability of Success of EVs, is 0.8 (80%)

Because both Launch Vehicles and Entry Vehicles are in early testing stages, true probability of mechanical failure will not be known for some time. In order to accurately present this aspect during analysis, a probability of failure of 0.2 (20%) is used for both vehicles; this results in a probability of success of 0.8 (80%). This estimate is based on the P_f of other unmanned space vehicles.

3.2 Problem Formulation

Input Data:

$P(S_{LV}(i))$ → Probability of Success of LV_i

$P(S_{EV}(i,j) \mid S_{LV}(i))$ → Conditional Probability of Success of LV_i's EV_j if LV_i succeeds

Decision Variables:

$P(S_f(i,k) \mid S_{EV}(i,j) \cap S_{LV}(i)) = f(F_{\text{Footprint}}(t,\theta)) \rightarrow$ Conditional Probability of Success dependant on location of Target$_k$ within the footprint of LV$_i$

$$x_{ijk} = \begin{cases} 1 \text{ when } \mathrm{EV}_{i,j} \text{ assigned to target k} \\ 0 \text{ otherwise} \end{cases} \rightarrow \text{Assignment Matrix}$$

Objective Function:

$$\max \sum_{k=1}^{\#\text{targets}} \left[\left(1 - \prod_{i=1}^{\#\text{LV}} \left[1 - \left(P(S_{LV}(i)) \left(1 - \prod_{j=1}^{\#\text{EV}} [1 - P(S_{EV}(i,j) \mid S_{LV}(i)) x_{ijk}] \right) P(S_f(i,k) \mid S_{EV}(i,j) \cap S_{LV}(i)) \right) \right] \right) W(k) \right] \quad (6)$$

Constraints:

$$\sum_{j=1}^{\#\text{EV}} \sum_{k=1}^{\#\text{targets}} x_{ijk} \leq 4 \ \forall\, i \in [1,2,\ldots,\#\text{LV}] \quad (7)$$

$$\sum_{k=1}^{\#\text{targets}} x_{ijk} \leq 1 \ \forall\, i \in [1,2,\ldots,\#\text{LV}],\ j \in [1,2,\ldots,\#\text{EV}] \quad (8)$$

Where:

i = LV #, j = EV #, k = Target #
W(k) = Weight of Target$_k$

Both the probability of success of Launch Vehicles and Entry Vehicles are assumed to be 0.8 (see Section 3.1). The decision variables for the assignment process include the assignment of LVs' EVs to targets, as well as the inclination and release time of these Entry Vehicles. The selection of inclination and release time affect the probability of success from a footprint (see Section 2.4.3), with $P(S_f \mid S_{EV} \cap S_{LV})$ a function of the inclination (θ) and release time (t). The matrix x_{ijk} is an assignment matrix, where each spot in the matrix correlates between a Launch Vehicle, Entry Vehicle, and target. The spot x_{ijk} is one when LV$_i$'s EV$_j$ is assigned to Target$_k$. The value of the objective function represents the expected value from a set of assignments. The objective function

(equation 6) includes both the probability of success from target location within a footprint and LV/EV combinations, as well as the weight of each target. The interior equations are reminiscent of the equations in Section 2.4.1 and 2.4.2, accounting for the portion of probability of success that comes from cross targeting, while the equation within the outermost curved bracket is for the total probability of success of Target_k. This is then multiplied by the weight of Target_k (resulting in an expected value for an assignment) and summed with all of the other expected values in order to get the total expected value. Equation 7 and 8 are constraints on the assignment matrix; equation 7 limits the number of targets assigned to each LV's EVs to four and equation 8 limits the number of targets each EV can be assigned to one. Equations 7 and 8 are based on assumptions. If the assumptions changed, the constraints would also change to match the assumptions.

3.3 Nonlinear Assignment Problem

Assignment problems are linear when the total utility for any combination of assignments is a linear combination of each assignment's utility. Assignment problems are considered nonlinear when the total utility does not equal the sum of the utility of each assignment. This effect takes place in problems where synergy occurs between assignments. A simple example would be the combination of ice cream and a banana. While they both have an independent utility, when combined together to make a banana split, their total combined utility is greater than the sum of their independent utilities. This type of synergy is present in the LV/EV assignment problem when cross-targeting occurs.

3.4 Network Flow Diagram

Network flow diagrams are a common and useful way to view assignment problems [11]. The following sections use network flow diagrams to illustrate the assignment process, and shed light on the effects of cross targeting on the utility function. Figure 3-1 is an image of a sample network flow diagram. This network flow diagram is for a two Launch Vehicle, four target problem. The two large blue circles on the left are the two Launch Vehicles. Each small green dot represents the assignment of an EV to a target. When all EV's from a given LV are assigned targets, the result appears as a "path" through the network. In the following sections, once an EV is assigned to a target, the

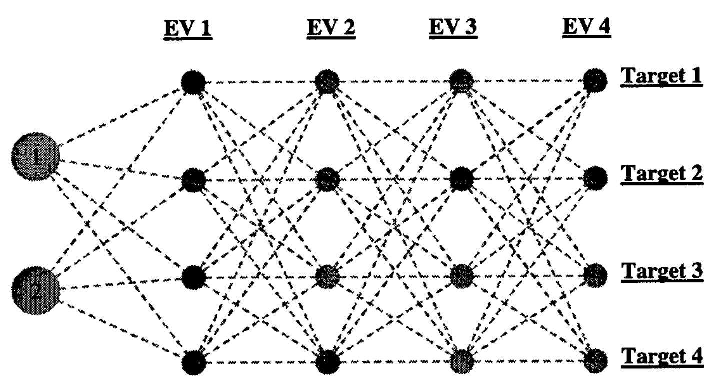


Figure 3-1. Network Flow Diagram for two Launch Vehicle, four Target problem

corresponding dot changes colors in increasing rounds to yellow, orange, and red to represent the number of times a target has been assigned an EV. When a number is listed next to a path, the number is the additional value of choosing that path (addition to the Objective function due to the new EV assignment). This diagram will be used in the subsequent sections to describe the network flow problem.

3.4.1 Basic Network Flow Breakdown

Although we view the assignment problem as a series of decisions in a network flow, in order to determine the incremental increase in utility from each assignment, the final utility is strictly a function of the combination of targets assigned to each LV. Figure 3-2 shows an example network flow diagram where two different paths result in the same assignment. In this case there are two different paths for the first Launch Vehicle, the blue solid line and red dashed line paths. Although these paths both take different routes through the network, they both result in the same target assignment, with two Entry Vehicles targeting the first target and two targeting the fourth target. Because both solutions will determine a best launch inclination and release time at the end of the

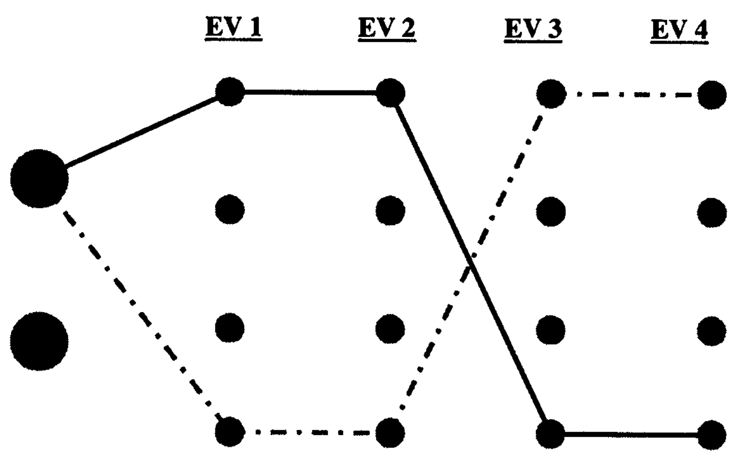


Figure 3-2. Network Flow showing path with identical results

results, both paths will result in the same solution. This points out the fact that some paths duplicate solutions given by other paths.

When other Launch Vehicles are included this flow becomes more difficult to quantify. Figure 3-3 gives an example of two sets of results. In this situation, the second Launch

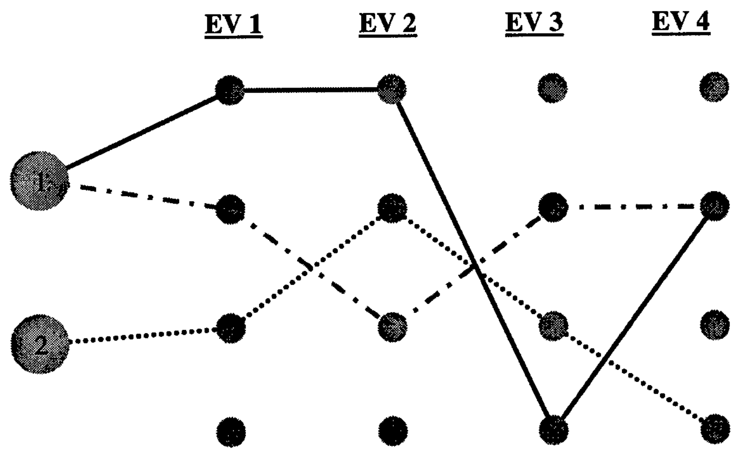


Figure 3-3. Network Flow showing expanded network from Launch Vehicle interaction

Vehicle has chosen the black path (dotted line) and the first Launch Vehicle has many different choices of paths, two of which include the blue (solid line) and red (dashed line) paths. Because the aggregate utility is a function all vehicles, the solution is a combination of every Launch Vehicle's path. Therefore, for each path through the network chosen by the second Launch Vehicle (dotted black line), there are numerous paths through the network for the first Launch Vehicle that lead to different utilities (solid blue or dashed red line). This combination of paths between Launch Vehicles causes the total number of solutions to increase exponentially with the inclusion of additional Launch Vehicles or targets. This results in problems with many Launch Vehicles, or many Targets, having very larger numbers of total solutions or path combinations.

3.4.2 Effects of Nonlinearities on Network Size

As defined in Section 3.3, nonlinearities occur in assignment problems when the combined utility of two assignments does not equal the sum of the assignments' utility. The nonlinearities in the assignment problem are a result of nonlinearities in the

probability of success due to cross targeting and target location in footprints. These effects were discussed in Section 2.4.1 through 2.4.4. The effect of each nonlinearity on the network flow is discussed in the subsequent sections.

3.4.2.1 Effect of Cross targeting with multiple Entry Vehicle from one Launch Vehicle

As discussed above, when targeting a location with more than one Entry Vehicle, the probability of success of a target does not increase linearly when targeted with additional EVs. This is discussed in great detail in Section 2.4.1. Because of this nonlinearity, the value of a path in the network flow is dependant on the result of earlier paths. An example of this can be seen in Figure 3-4. Here, two separate paths are taken through the

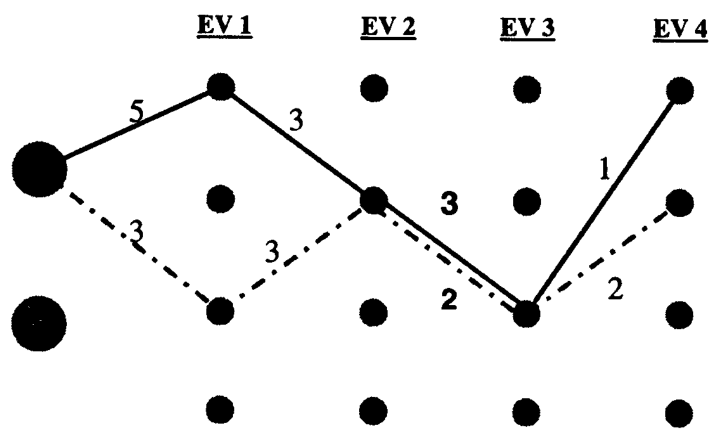


Figure 3-4. Network Flow with nonlinearity from Cross Targeting from one Launch Vehicle

flow; the first is represented by the blue solid line while the second is represented by the red dashed line. Although these paths are separate for much of the process, both assign the same targets to their second and third Entry Vehicles.

Note that the path from the second to third Entry Vehicle is worth three to the blue solid path, but two to the red dashed path because the path had already targeted this target. As shown in Section 2.4.1, targeting a location with multiple EVs increases the P_S of destroying the target, but in decreasing increments. The red dashed line between the second and third Entry Vehicles adds two to the path utility, resulting in a total of five added utility from targeting the third target (three from first EV, two from third EV). Although this increases the utility from the target, hence the P_S of destroying the target, it increases in decreasing increments. Although the values of a path can be added to determine total utility, the value of a path is dependant on earlier assignments along a path.

3.4.2.2 Effect of Cross targeting with multiple Entry Vehicles from multiple Launch Vehicles

A similar effect to the one discussed in Section 3.4.2.1 results from the synergistic effect of cross targeting locations with Entry Vehicles from multiple Launch Vehicles. When a location is already targeted by EVs from one LV, the added utility of targeting the location again with an EV from a different Launch Vehicle is dependant on the increase in the P_S. This value is dependant on how many Entry Vehicles are targeting a location, and which Launch Vehicles they are released from. Figure 3-5 displays a network flow where multiple EVs from multiple LVs target the same location. In the diagram, the first Launch Vehicle has already chosen either the solid blue or dashed red path; then the second Launch Vehicle chooses the dotted black path. However, the value added to the

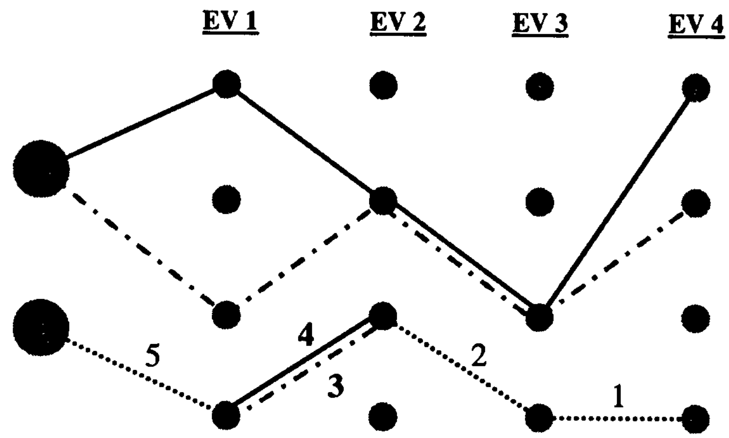


Figure 3-5. Network Flow with nonlinearity from Cross Targeting from multiple Launch Vehicles

objective function by choosing this path is dependant on the other paths. In this case, both Launch Vehicles have chosen to target the third target with at least one Entry Vehicle. Because this target is targeted by EVs from multiple LVs, the added value of choosing this target is dependant on how many times it has been targeted by other Launch Vehicles. If the first Launch Vehicle chooses the solid blue path the third target will have been targeted once, while choosing the dashed red path will result in it being targeted twice. Because of this, the added value of the third target for the second Launch Vehicle is higher for the solid blue path (four) than the dashed red path (three). This occurs any time a target is cross targeted by EVs from multiple Launch Vehicles which, due to the benefits of increased probability of success from cross targeting addressed in Section 2.4.2, occurs frequently in assignment problems.

3.4.2.3 Nonlinearities from Launch Inclination and Release Time

Figure 3-4 and Figure 3-5 show values associated with each branch of a path. These values are based on the value added to each path based on the objective function when

the decision represented by this path is made. Part of this objective function is a probability of success of destroying a target associated the location of a target within a footprint. This is discussed in Section 2.4.3. Since all Entry Vehicles of a Launch Vehicle are released at the same time, all share the same footprint. This results in footprint selection affecting all utilities along a path. Because of this, each different release time and inclination results in a different value for a path. This is shown in Figure 3-6, where three paths (solid blue, dashed red, and dotted black) all go though the same points, but have different values. The Footprints associated with the release time and

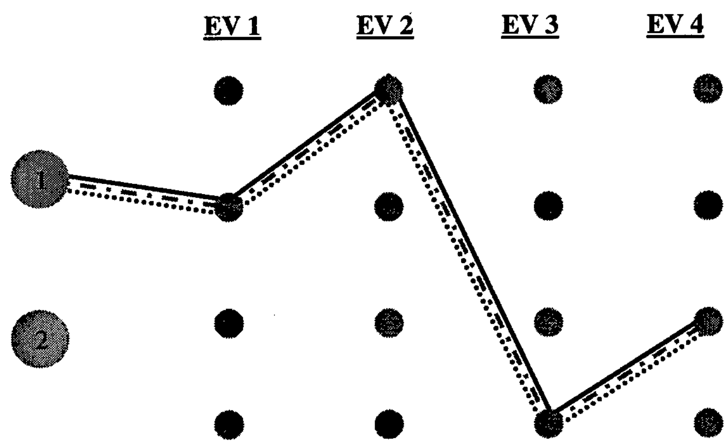


Figure 3-6. Network Flow with nonlinearity from Footprints

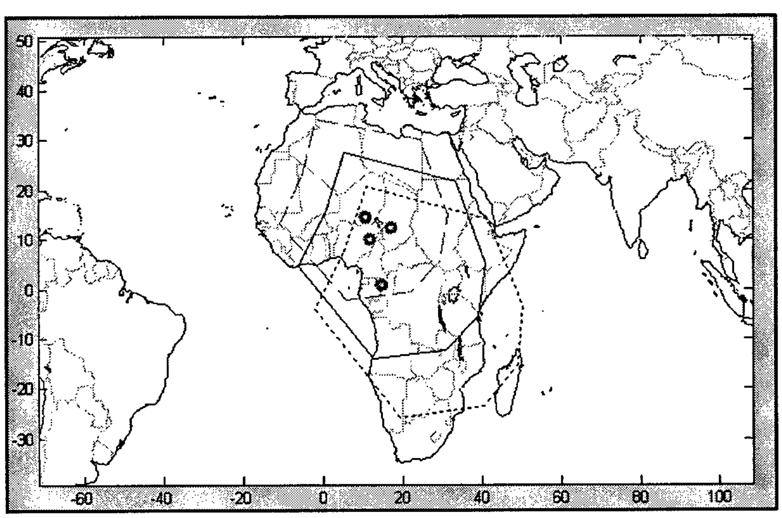


Figure 3-7. Three Footprints associated with paths in Figure 3-6

inclination of each path can be seen in Figure 3-7, where the colored footprints relate to the colored paths. In order to calculate the added utility to the objective function of a step between two points, a footprint must be chosen at each point along a path. The added utility of each point is dependant on whether a footprint changes from point to point. If the same inclination and release time is chosen, the added utility of the path is dependanton changes to the $P(S_{EV} \cap S_{LV})$. However, if different values are chosen, the entire value of the path to the last point is changed. This causes the value of the new path to be the change in value of the previous path plus the increase in $P(S_{EV} \cap S_{LV})$ due to targeting.

Because the value of each path is dependant on the launch inclination and release time of each path, a path actually represents numerous paths, each with a different release time and inclination. This would result in a multi-dimensional network flow diagram; however, since the release time and inclination is easily optimized to maximize the added utility by each path, only one release time and inclination will be considered for each path.

4 Solution Method

To solve the nonlinear assignment problem, this research proposes a solution methodology that involves breaking the problem into smaller assignment problems that can be solved using linear methods. Categorizing solution methods by the way they assign targets to Launch Vehicles, two basic methods are discussed: Breadth and Depth based methods. Each method involves the same process to calculate utilities, but a different order in which targets are assigned. One version of the Breadth based method uses Linear Auctions to reach optimal solutions for each cycle of the Breadth based method. These solutions can be improved over time by use of the Directed Search, which eliminates poor decisions. Another viable use of the Breadth Linear Auction is through the use of random weight perturbations, which runs numerous Breadth Linear Auctions with Random weight perturbations in an attempt to improve on the assignment. Finally, two methods used to compare results from the solution methods are discussed.

4.1 Comparison of Breadth vs. Depth

When searching a network for a solution, search methods can be categorized into two basic categories: breadth-first and depth-first searches [11]. A breadth-first search involves searching all nodes closest to the starting node, then expanding out to search nodes farther away [10]. This correlates to assigning targets to each Launch Vehicle's first Entry Vehicle, then each Launch Vehicle's second entry Vehicle, etc., until all assignments have been made. From a network flow perspective, the assignments are made by columns, making all first column assignments, then second, etc. A network

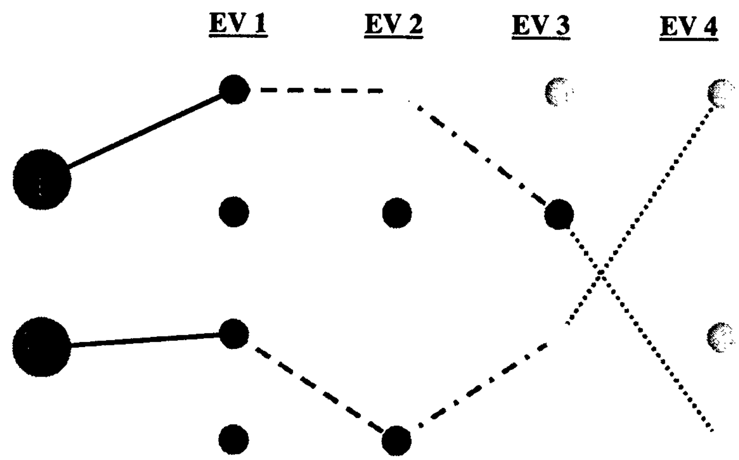


Figure 4-1. Network Flow showing assignment via Breadth based Methods

flow representing this is shown in Figure 4-1, where the first assignments made are the blue solid assignments, then the red dashed, the green dashed and dotted, and finally the magenta dotted. A depth-first search involves searching a path of nodes to the solution, then searching other paths to the solution [10]. For the assignment problem, this relates to deep search into the network first, which results in obtaining more information about deeper parts of the network. As opposed to Breadth based methods, Depth based methods assign targets to each of the first Launch Vehicle's Entry Vehicles, then each of the second Launch Vehicle's Entry Vehicles, etc., until all assignments have been made.

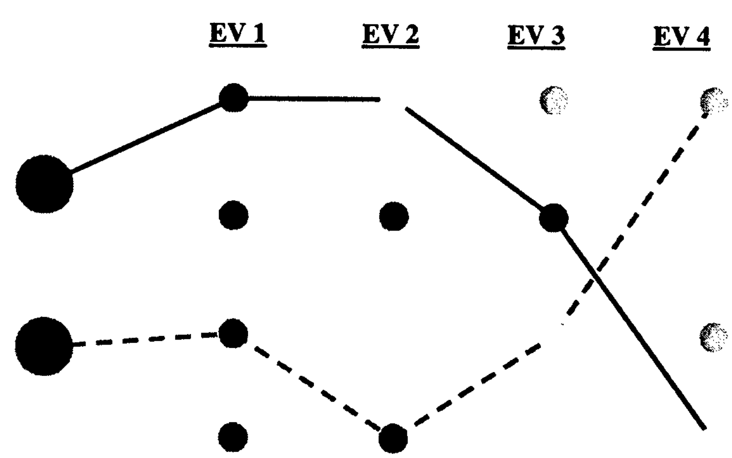


Figure 4-2. Network Flow showing assignment via Depth based Methods

Figure 4-2 displays a network flow diagram showing the sequence of assignments made via a Depth based method. The blue solid line represents all of the assignments made first, by the first Launch Vehicle, while the red dashed line represents all of the assignments, for the second Launch Vehicle, made second. Since all assignments are made for the First Launch vehicle before any are made for the second Launch vehicle, this method provides the second Launch vehicle with complete knowledge of assignments by the first Launch Vehicle during each of its assignments.

4.2 Utility Determination Process

For this problem, both the Breadth and Depth based methods use the same process to determine the utility of an assignment. This process is used to determine the added utility of each incremental step (assignment) in the process. Although both assignment methods will be discussed in subsequent sections, it is important to give a description of how each utility is calculated. Figure 4-3 gives a graphical representation of the process of determining utilities for assignments. The process begins with the left most block and cycles though until all processes are complete. The first step during each cycle of the process is to determine the current value of the objective function. At the beginning of

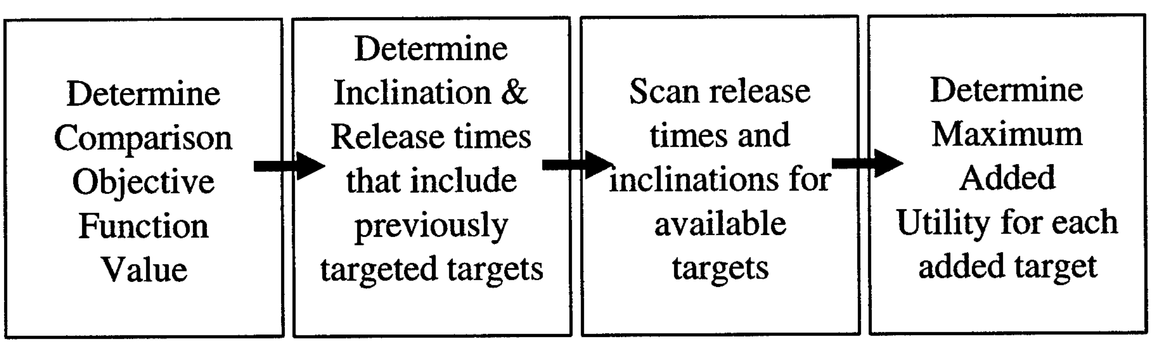


Figure 4-3. Utility Development Process Diagram

each method, when no assignments have been made, this value is zero. In subsequent cycles this will be determined by calculating the value of equation 6 due to the current assignments, launch inclinations, and release times for each LV. This value will be used in subsequent steps as a comparison to determine the change in the objective function due to additional assignments.

The second step is to determine all inclinations and release times that include previously assigned targets for each Launch Vehicle. This is done by examining the $P(S_f \mid S_{EV} \cap S_{LV})$ chart discussed in Section 2.4.3.1. During the first round, since no assignments have been made, all inclinations and release times are viable for assignments. However, in subsequent rounds there will be less viable choices. For example, if a Launch Vehicle has already been assigned target two in a previous cycle, the only release times for the inclination represented in Table 4-1 that result in a non-zero

Table 4-1. $P(S_f|S_{EV} \cap S_{LV})$ chart with Highlighted Viable Release Times

$P(S_f\|S_{EV} \cap S_{LV})$ values for one Inclination					
		Release Time			
		t1	t2	t3	t4
Target	1	0	0.1	0	0
	2	0	0.2	0.3	0.3
	3	0	0	0.25	0.7
	4	0.25	0.5	0.25	0

$P(S_f \mid S_{EV} \cap S_{LV})$ for target two are t2, t3, and t4 (highlighted). These release times will be used in the next process to determine which targets are available to be targeted. The third step in the process is to determine which targets are available for assignment. This process is limited to targets with positive $P(S_f \mid S_{EV} \cap S_{LV})$ values in inclinations and

release times which include previously assigned targets. This is done to ensure that whichever target is assigned during the current process, an inclination and release time exists that has positive $P(S_f \mid S_{EV} \cap S_{LV})$ values for all assigned targets. During the first round, since no assignments have been made and all inclinations and release times are available, any target can be assigned to any vehicle. Continuing the example from the previous paragraph, using Table 4-1 to determine available release times, all four targets may be assigned because all have positive $P(S_f \mid S_{EV} \cap S_{LV})$ values for the highlighted release times. However, if the Launch Vehicle selects the first target during this cycle of the assignment process, release times would be limited to t2 in subsequent cycles because it is the only release time with positive $P(S_f \mid S_{EV} \cap S_{LV})$ values for both the first and second targets. This would limit targeting in subsequent rounds to targets one, two, and four.

The fourth step in the process is to determine the increase in the objective function value for each additional assigned target. This value will be termed the utility value, representing the value or utility of a choice. To determine the utility value a simple brute force search is completed comparing the new objective function value to the value calculated in the first step for each available target in each release time in each inclination. The maximum value for each additional target, based on the best release time and inclination, is recorded as the utility value of that target. This also results in the determination of a release time and inclination corresponding to each target which will become the new inclination and release time following target assignment. Although brute force may not appear to be a very effective way to determine these values, due to

the small number of calculations and the small amount of time each calculation takes, the method is adequate. The size of this search, and length of time needed to complete the process, is directly linked to the number of release times and inclinations that need to be searched.

4.3 Breadth Methods

By definition, breadth is a wide range or scope. For Breadth based methods, the goal is to have the greatest amount of knowledge about the decisions of other Launch Vehicles when making the decision for the current Launch Vehicle [10]. The subsequent sections explain a basic implementation of the Breadth method. The use of a Linear Auction for assignments in a Breadth method is discussed in Section 4.5.

4.3.1 Breadth Greedy

The Breadth Greedy method is the most basic implementation of the Breadth method, assigning targets based on a Greedy methodology [10]. The Breadth greedy method uses four cycles of the utility development process, one for each column of Entry Vehicles. During each cycle of utility development, the Breadth Greedy method develops utilities for all Launch Vehicles based on assignments of all previous columns (all previous cycles). Once utilities are developed for each Launch Vehicle, the Breadth Greedy Method begins with the first LV and chooses the assignment which will maximize its utility [10]. It then proceeds to the next Launch Vehicle, maximizing each Launch Vehicle's utility. Due to the breadth limitation that only one Launch Vehicle may be

assigned to a target during each cycle (column) of assignments, Launch Vehicles that make assignments first will generally result in larger utility values, while Launch Vehicles that make assignments later will generally obtain lower utility values due to a limited number of targets still available for assignment. This basic method will be used as a comparison to analyze the effectiveness of the use of Linear Auctions to make assignments within a Breadth based method.

4.3.2 Strengths of Breadth Methods

The major strength of all Breadth Methods is the ability to take assignments from earlier cycles into consideration when making new assignments. The ability to consider previous assignments when making new assignments allows for accurate accounting of probability of success. The breadth method also allows for the calculation of utility for numerous Entry Vehicles during the same cycle, which allows for the implementation of linear optimization techniques during each cycle of a Breadth based method to optimize the increase in the objective function per cycle.

4.3.3 Weaknesses of Breadth Methods

The major weakness of Breadth-first methods is the lack of information when making first Entry Vehicle assignments. Since all Launch Vehicles make their assignments for their first Entry Vehicle during the same cycle, they do not have any knowledge of how the selection will limit future assignments. Because the first assignment limits possible inclinations and release times for future assignments, if a poor first assignment is made it

may limit the final value of the Objective Function. This can occur when highly weighted targets are grouped, resulting in all Launch Vehicles limiting release times and inclinations to this group, eliminating the ability to target locations outside of the group. An example of this is given in Section 5.1.3.

4.4 Depth Methods

Depth methods, as opposed to Breadth methods, attempt to search deep parts of the network quickly in order to make better decisions in the assignment of targets to early Entry Vehicles for Launch Vehicles that make their assignments after the first Launch Vehicle [10]. This results in a tradeoff in improvements. While this may result in better selection of assignments for earlier Launch vehicles, it will also result in lower utility values for later Launch Vehicles. This can cause later Launch Vehicles to result in extremely low added utility, resulting in lower performance.

4.4.1 Depth Greedy

The Depth Greedy method, similar to the Breadth Greedy, uses numerous cycles of the utility determination and the Greedy methodology to make assignments. However, as opposed to the Breadth Greedy, which uses four large cycles each resulting in an assignment of one Entry Vehicle for each Launch Vehicle, the Depth Greedy uses four large cycles each resulting in an assignment of every Entry Vehicle for one Launch Vehicle, each with four sub cycles. Each sub cycle follows one path though the utility determination process with reference to one Launch Vehicle's Entry Vehicle. Figure 4-4

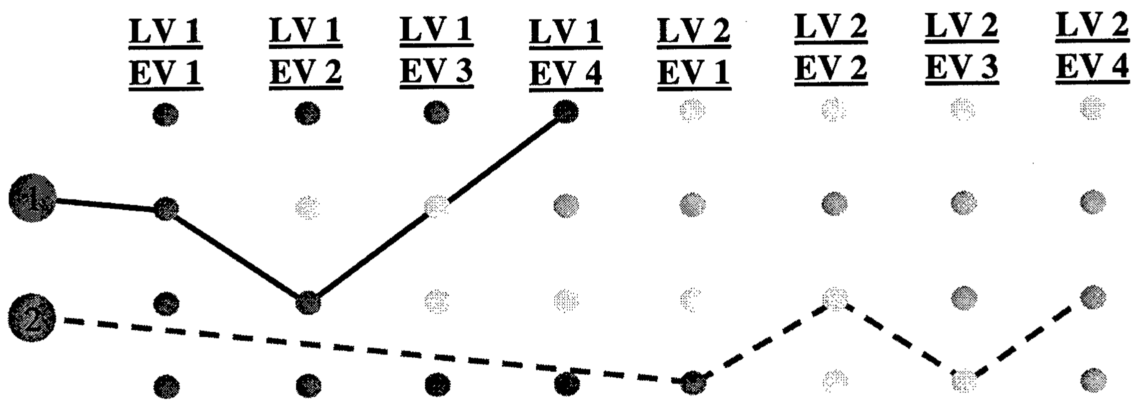


Figure 4-4. Network Flow for Depth Greedy Method

shows a network flow of all assignments made during each sub cycle of the Depth Greedy method for a two Launch Vehicle, four target problem. The first four columns represent the four sub cycles (four Entry Vehicles) of the first cycle (first Launch Vehicle), where assignments are made for each of the first Launch Vehicle's Entry Vehicles. The change in target color shows that when the second Launch Vehicle is making an assignment for its first Entry Vehicle, it has knowledge that the first target has not been targeted, while the first and third have been targeted once, and the second has been targeted twice.

4.4.2 Strengths of Depth Methods

The main strength of Depth based methods is an improvement in the selection of an assignment for each Launch Vehicle's first Entry Vehicle. Because the first Entry Vehicle assignment, for Launch Vehicles other than the first Launch vehicle, is chosen following the assignment of all previous Launch Vehicles' Entry Vehicles, these Launch Vehicles have more information about which targets have and have not been targeted, and can avoid reassigning targets assigned to earlier Launch Vehicles. This allows

Launch Vehicles to avoid selecting targets which will result in a release time and inclination that will overlap with other Launch Vehicles' release times and inclinations. This avoids forcing all EVs to target the same set of targets, and allows dispersion to other targets.

4.4.3 Weaknesses of Depth Methods

The main weakness of Depth based methods is the distribution of utilities. Because the first Launch Vehicle assigns targets to all EVs before subsequent LVs assign any to their EVs, the first Launch Vehicle should have the highest utility, while the second will have the second highest, etc. Although these utilities are higher than those in Breadth based methods, they may force later Launch Vehicles into selections that result in very low utilities, resulting in an overall lower objective function result. An example of this is given in Section 5.1.2. Also, it becomes much more difficult to implement linear optimization techniques in Depth methods. This is discussed in section 4.5.9.

4.5 Linear Auction

Linear Auctions are used to determine assignments for linear assignment problems. Similar to their economic counterpart, Linear Auctions serve to find the optimal solution to a linear assignment problem by breaking the problem into bidders (Launch Vehicles) and items (targets), allowing the bidders to reach an equilibrium assignment which results in the optimal assignment of bidders to items. This section includes a background on

Auctions, discussion on the linear auction process, and how a Linear Auction can be used to make assignments in a breadth based solution method.

4.5.1 Auction Background

Auctions have been an area of interest since the 1960's due to their use in the economic market as a means of obtaining an optimal value for a given item based on market valuation [27]. This research includes a comparison between different types of auctions, and auction mechanisms, in an attempt to increase the value gained via auctions [17]. This interest has led to numerous government auctions for goods with a natural monopoly, the most notorious being the governments use of auction to sell radio rights via the FCC [9]. This same capability led to its introduction into assignment problems, where the given parties have the same desires as those in traditional auctions. In order to apply these capabilities, a methodology to implement numerical auction algorithms was developed [1].

Although there are many ways to solve linear assignment problems, Linear Auctions have shown to be a very powerful method featuring short worst case solution times and internal knobs which allow tradeoffs between solution accuracy and speed. Two other comparable solution methods include the simplex (primal descent) and primal-dual (dual ascent) methods. Compared to the linear auction method these have large worst case solution times; the simplex method has an exponential worst case time, $O(c^n)$ where c is a constant and n is the number of items in the problem, and the primal-dual has a pseudo polynomial worst case time [12]. The auction method, however, can be shown to have a

polynomial worst case time, O(nmLog[c*n]) where n is the number of bidder and m is the number of items [12]. While this does not guarantee that an auction will reach a solution faster, it does guarantee that for problems where all methods take the maximum amount of time to reach a solution, Auctions will reach a solution quickest.

4.5.2 Auction Process Overview

The formulation of an auction includes a group of bidders (i), a group of items (j) to be bid on, a starting price (p_j) for each item, and a perceived utility for each item for each bidder (a_{ij}). Auctions can start with assignments for each bidder, but generally start with no assignments. The starting price for each item can be any value; however, the closer starting values are to final values, the quicker a solution will be reached. Once the auction begins, a sequence of bids is obtained from bidders. Each bidder places "his" bid on the item with the maximum perceived value (v_{ij}) (i.e. difference between the utility to

$$v_{ij} = a_{ij} - p_j \tag{9}$$

the bidder and current cost). The bid (Bid_{ij}) is based on the current price of the item, and the difference between the best value (v_{ij}) and the second best value (ω_{ij}) for an item.

$$v_{ij} = \max \{a_{ij} - p_j\} \tag{10}$$

$$\omega_{ik} = \max_{k \neq j} \{a_{ik} - p_k\} \tag{11}$$

$$\gamma_{ik} = v_{ik} - \omega_{ik} \tag{12}$$

$$Bid_{ij} = p_j + \gamma_{ik} + \varepsilon \tag{13}$$

This is done to eliminate extra cycles, forcing the bidder to bid the maximum amount for an item which allows it to remain the most valuable item. This bid also includes a bid

increment (ε), which is both a tool used to ensure convergence and a knob to allow a tradeoff between solution time and solution accuracy; bid increments are discussed in greater detail in Section 4.5.5. This process continues until each bidder has placed a bid which is the highest bid placed for an item (i.e. auction prices reach an equilibrium). The final assignment is made between items and the bidders which placed the highest bid for the respective item. An example of this process is discussed in Section 4.5.3.

4.5.3 Example Linear Auction

In order to best explain the benefits of a linear auction, it is good to look at the results of a simple assignment problem by two means. Figure 4-5 shows the result using a Greedy solution method and a Linear Auction. The first solution (Figure 4-5a) is a greedy solution where the bidder with the most utility gets the first choice, and the bidder with the second most the next available choice, etc. This results in a non-optimal solution, where the combined value of the three highlighted blocks is 35. The second solution (Figure 4-5b) is solved using a linear auction. Unlike the first solution method, the goal

Table of Utilites

		Bidder #		
		1	2	3
Item #	1	20	15	10
	2	18	10	8
	3	10	8	5

(a)

Table of Utilites

		Bidder #		
		1	2	3
Item #	1	20	15	10
	2	18	10	8
	3	10	8	5

(b)

Figure 4-5. Comparison results between Greedy and Auction

of this method is to result in the best overall utility, which in this case is 38. This improvement shows the benefit to an individual cycle of the Breadth Method by using a

Linear Auction rather than Greedy means to make assignments. In order to better explain the actual workings of a linear auction, a round by round example is given in Section 4.5.4.

4.5.4 Round by Round Example Linear Auction

Auction Round #	1		
Bidder	1		
Items	1	2	3
Starting Prices	15	10	5
Utilites	20	18	10
Profit	5	8	5
(Profit = Utility - Price)			
Best Object, Profit	2	8	
2nd Best Object, Profit	1	5	
Difference		3	
Object Bid On, Bid	2	14	
(Bid = Start Price + Difference + Bid Increment)			
Ending Prices	15	14	5

Assignment Start		
Object #		
1	2	3
Bidder #		
End		
Object #		
1	2	3
Bidder #		
	1	

Assignment Start		
Bidder #		
1	2	3
Object #		
End		
Bidder #		
1	2	3
Object #		
2		

Figure 4-6. Results of 1st Round of Example Linear Auction

In order to ensure that that auction takes a small number of rounds to reach a solution, starting prices for the three items were set at 15, 10, and 5 respectively. During the first round of the auction, Bidder one compares its utility to the prices for each item, determining profits for each item. Once this is done, bidder one compares the profit from the most profitable item (8 for item two) and the next most profitable item (5 for item one). This is done to ensure that the bid placed on the highest valued item is the largest possible bid that allows this item to remain the most valuable. Once this is completed, bidder number one places a bid of 14 on item number two. This is determined by adding the starting price (10), the difference between the most profitable and next most profitable item (3), and the bid increment (1).

Auction Round #	2		
Bidder	2		
Items	1	2	3
Starting Prices	15	14	5
Utilites	15	10	8
Profit	0	-4	3
(Profit = Utility - Price)			
Best Object, Profit	3	3	
2nd Best Object, Profit	1	0	
Difference		3	
Object Bid On, Bid	3	9	
(Bid = Start Price + Difference + Bid Increment)			
Ending Prices	15	14	9

Assignment		
Start		
Object #		
1	2	3
Bidder #		
	1	
End		
Object #		
1	2	3
Bidder #		
	1	2

Assignment		
Start		
Bidder #		
1	2	3
Object #		
2		
End		
Bidder #		
1	2	3
Object #		
2	3	

Figure 4-7. Results of 2nd Round of Example Linear Auction

Once this is completed, round two begins. This round will determine which item bidder two should bid on. The starting prices match the ending prices of the previous Round, and the starting assignments also match the ending assignments from the previous round. This round follows the same procedure as the previous round, and results in bidder two placing a bid of 9 on item three.

Auction Round #	3		
Bidder	3		
Items	1	2	3
Starting Prices	15	14	9
Utilites	10	8	5
Profit	-5	-6	-4
(Profit = Utility - Price)			
Best Object, Profit	3	-4	
2nd Best Object, Profit	1	-5	
Difference		1	
Object Bid On, Bid	3	11	
(Bid = Start Price + Difference + Bid Increment)			
Ending Prices	15	14	11

Assignment		
Start		
Object #		
1	2	3
Bidder #		
	1	2
End		
Object #		
1	2	3
Bidder #		
	1	3

Assignment		
Start		
Bidder #		
1	2	3
Object #		
2	3	
End		
Bidder #		
1	2	3
Object #		
2		3

Figure 4-8. Results of 3rd Round of Example Linear Auction

These results lead to round three, where bidder three bids. Two major changes occur in this round. First, bidder three's valuation for each item is less than its current price.

While common sense may seem to dictate that bidder three should not bid on any items, because each bidder must be assigned to an item at the end of the auction, bidder three must bid on the item that will cause it to lose the least amount, hence item three. This leads to the second change, that bidder three chooses to bid on an item already assigned to bidder two. Because bidder three outbids bidder two, it is the winner of the item at the end of the round, resulting in a change in assignment at the end of the round.

Auction Round #	4		
Bidder	2		
Items	1	2	3
Starting Prices	15	14	11
Utilites	15	10	8
Profit	0	-4	-3
(Profit = Utility - Price)			
Best Object, Profit	1	0	
2nd Best Object, Profit	3	-3	
Difference		3	
Object Bid On, Bid	1	19	
(Bid = Start Price + Difference + Bid Increment)			
Ending Prices	19	14	11

Assignment Start		
Object #		
1	2	3
Bidder #		
	1	3
End		
Object #		
1	2	3
Bidder #		
2	1	3

Assignment Start		
Bidder #		
1	2	3
Object #		
2		3
End		
Bidder #		
1	2	3
Object #		
2	1	3

Figure 4-9. Results of 4th Round of Example Linear Auction

This leads to the final round, where bidder two bids again. As opposed to the first three rounds where the bidder was chosen in sequence, in round four and future rounds, the bidder is chosen based on whom has gone the longest (most rounds) without an assignment. Because the second bidder last had an assignment at the end of the second round, "he" has gone one round without an assignment. Since all other bidders currently have assignments, bidder two has gone the longest (most rounds) without an assignment. During the fourth round, bidder two bids on item one. Because at the end of the fourth round each bidder has been assigned an item, the auction ends.

4.5.4.1 Alternative Parallel Method

Although the auction example in Section 4.5.4 displays one bidder bidding during each round, it is possible to run the auction with numerous bidders bidding during each round. In such a variation, all bidders that are not currently assigned to an item would complete the bidding process during each round. At the end of the round, an auction manager would determine which bids are the highest, making new assignments, and continuing to cycle through the process. Although this type of auction can show significant increases in solution speed when calculated on a parallel computer [1], the process does not reach a solution faster on a serial computer, and can actually reach solutions slower than the method discussed in Section 4.5.4. Therefore, the serial auction method discussed in Section 4.5.4 was used in all calculations rather than the parallel method. However, if parallel computers are available, the parallel method should be implemented to increase solution speed.

4.5.5 Bid Increment

One very important part of an Auction Algorithm is the bid increment (ε). The bid increment serves a dual purpose as a tool to ensure convergence to a solution as well as an internal knob between solution time and quality. Without a bid increment, many auctions would converge because of the setup of different utilities between bidders. However, there are situations where the right difference in utility would exist between two bidders that would cause an auction to not converge. Without a bid increment, if a situation occurred where the relative difference between the most valuable items was

zero, a bidder would only bid the current price for an item. If another bidder had the same relative difference between the two items, this bidder would also bid the current price for the item. This cycle would continue forever, causing the auction to not converge. The bid increment avoids this by forcing each bidder to increase the price of the item they are bidding on, or essentially overbid an item, by the amount of a bid increment.

The bid increment also serves as an internal knob. Using large bid increments results in a faster solution, but more errors; smaller bid increments result in better accuracy, but slower solution times. The Linear auction method guarantees an optimal solution to a linear assignment problem when the equation 14 is true, where n is the number of

$$\varepsilon \cdot n = 1 \tag{14}$$

bidders [12]. This can be simply accomplished by decreasing the bid increment to 1/n. However, by decreasing the bid increment, the auction can take much longer to reach a solution. A method to decrease the solution time, yet maintain an optimal solution, is discussed in Section 4.5.7

4.5.6 Linear Auction Convergence

In order for a linear auction to guarantee convergence two things must be present: a bid increment and a possible solution. As mentioned in Section 4.5.5, a bid increment is used to ensure that a linear auction converges to a solution. By forcing a bidder to overbid an item by a bid increment, the linear auction ensures that two bidders with the same relative valuation for the same items will bid and be assigned to one of the two items. By

overbidding on the item, the other item becomes more desirable to the other bidder, allowing it to bid and be assigned to the other item.

Although the presence of an actual solution may seem obvious, it is often more difficult to decipher. Effectively, this requires there to be a viable solution for an assignment problem where each item is assigned to a bidder. Although this will always occur in problems such as the example in Section 4.5.3, for problems where bidders do not have a utility for certain items (i.e. negative infinity utility) situations can occur where there are more bidders that have utilities for items in a group than there are items in the group. In these situations, the Linear Auction will not converge since the requirement for reaching a solution is for all bidders to have an assignment. Because of this it is important to analyze the behavior of the Breadth Linear Auction to determine if a situation could arise where a Linear Auction would not converge.

For this problem, the easiest way to guarantee a viable solution is to ensure that each Launch Vehicle has at least one assignable target that isn't the only assignable target of any other Launch Vehicle. For each Linear Auction within a Breadth Linear Auction, this will always occur because of the ability for a Launch Vehicle to target the same location with multiple Entry Vehicles. When a round ends, each Launch Vehicle is assigned a target, and hence an inclination and release time that will allow it to hit this and all other targets that have positive $P(S_f \mid S_{EV} \cap S_{LV})$ values for that release time and inclination. Because each Launch Vehicle must be assigned to a different Entry Vehicle during each cycle (column of assignments) of a breadth linear auction, at the beginning of

a cycle each Launch Vehicle currently has an inclination and release time that allows it to target all assigned targets. Because of this, the simplest solution involves each Launch Vehicle assigning all of its Entry Vehicles to the same targets. Because this is a viable solution, convergence is guaranteed.

4.5.7 ε–Scaling

Following the method discussed in Section 4.5.4, a linear auction is within $\varepsilon * n$ of optimality, where ε is the bid increment and n is the number of bidders. In order to ensure optimality, the bid increment should be reduced to ensure that equation 14 is met. Doing this, however, increases the number of rounds to reach a solution; hence the amount of time the auction takes to complete make assignments. For large bidder problems, this can drastically increase the auction running time. In order to avoid long running time, better starting prices for items must be chosen. However, they must not be chosen arbitrarily, but based on the bidder's utilities for items. In order to do this successfully, a series of auctions must be run, starting with a large ε value, and decreasing until reaching a ε that meets equation 14; this method is known as ε–scaling. Following this method, the first auction runs with a starting price of zero for all items. Each subsequent auction in the series starts with a starting price equal to the ending prices at the end of the previous auction. The increment chosen in decreasing ε values also determines the number of auctions that must take place in series to result in the final optimal solution; for simplicity, ε can be changed in 10x increment (ex. 1000, 100, 10, 1, etc.), with a final ε value that meets equation 14.

4.5.8 Breadth Linear Auction

The Breadth Linear Auction follows the same use of the Breadth-first methodology as the Breadth Greedy method, except assignments are selected though use of a sequence of Linear Auctions rather than by greedy means. The Breadth Linear Auction completes four cycles of the Utility Development process, with each cycle calculating utility for an Entry Vehicle for each Launch Vehicle. The assignment of targets to LVs is completed though the use of a series of Linear Auctions (as discussed in Section 4.5.7), resulting in an optimal assignment for each cycle (maximum added utility).

$$\sum_{i=1}^{\#LV}\sum_{j=1}^{\#EV} x_{ijk} \leq 4 \ \forall \ k \in [1,2,\ldots,\#\,\text{targets}] \tag{15}$$

Because only one Launch Vehicle may bid and win an assignment for a target during each auction, a maximum of four Entry Vehicles may be assigned to a target. This acts as an additional constraint, shown in equation 15. Although this limits the P_S for a target, this still results in a relatively large probability of success. Table 2-1 shows that the lowest probability of success from cross targeting with four EVs is a $P(S_{EV} \cap S_{LV})$ of 0.799, with a maximum of 0.983. Therefore, although targeting with additional EVs will increase the P_S, the diminishing returns are small enough that substantial changes to the Objective Function are not seen. This method should always result in an improved solution over the Greedy Breadth method, and will be used heavily in the Directed Search (Cut) method in an attempt to improve on assignments by iterating on target assignments. This method is discussed in Section 4.6.

4.5.9 Hybrid Method

Because a Linear Auction requires a group of bidders and items, the only way to implement a Linear Auction in a depth-first method is through the auctioning of groups of targets (auction targets for all Launch Vehicles); this would allow the bidders to be different Launch Vehicles while items for bid to be groups of targets. While this method is often used to solve nonlinear assignment problems, when the number of groups becomes large it is computationally impossible to solve the problem; this is the case for the target assignment problem. Therefore, because it is impossible to implement a Linear Auction into a depth based method, a hybrid was developed which uses Depth based methods to make the assignments for the first Entry Vehicle for each Launch Vehicle, and a Linear Auction to make the assignments for all other Entry Vehicles. This is accomplished by completing a single Depth Greedy assignment process for a problem, keeping the assignments for each Launch Vehicle's first Entry Vehicle, and then running the Breadth Linear Auction to make the assignments for the remaining Entry Vehicles. This method could show improvement of results over the Breadth Linear Auction method, but will take considerably longer to reach a solution (will have to complete two assignment methods rather than one).

4.6 Directed Search (Cut) Method

Although the Breadth and Depth based methods presented result in viable solutions to the assignment problem, none of the methods can guarantee an optimal solution. Similarly, although they are good solution methods, once they are run and processed once, no

further increases in the Objective Function can be achieved through the methods. In order to improve upon these solutions, a method is used to determine poor path selection. This allows the process to be repeated, avoiding the limiting paths. The Directed Search method is also referred to as the cut method because eliminating poor choices effectively prunes or cuts a branch off of the search path, eliminating poor results.

4.6.1 Directed Search Process

The Directed Search Method follows four basic steps, and repeats until either the user decides to stop the process, or all available solution paths have been eliminated. A graphical representation of the methodology is show in Figure 4-10, where the process

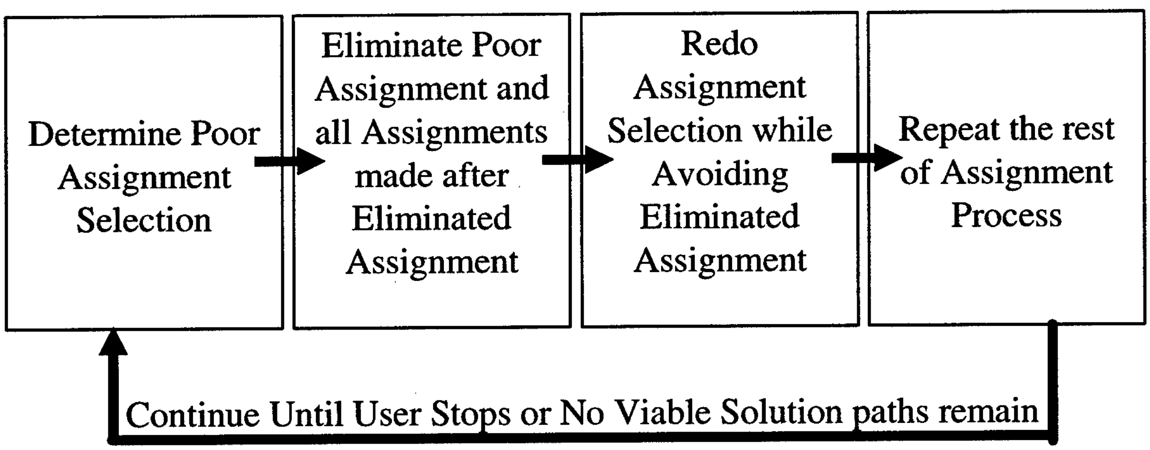


Figure 4-10. Directed Search (Cut) Process diagram

begins in the left block and continues clockwise until the end. The first step in the process is to determine the poor assignment selection. This process is relatively complicated and is described in more detail in Section 4.6.2. Once an assignment has been chosen to be eliminated, all assignments that were made after that assignment are eliminated as well as the assignment itself. The third step in the process leads to new

assignments for all eliminated assignments. These assignments are determined using the Breadth Linear Auction assignment method. Once this is completed, the process repeats itself. Numerous network flow diagrams in Section 4.6.3 show how numerous cycles of the Directed Search method limit the search space.

4.6.2 Cut Choices

The method used to determine which assignment choice is eliminated, or cut, is based on the principle that each Launch Vehicle should add approximately the same amount to the overall Objective function. Although this principle does not function with Greedy methods because their nature causes a disparity between Launch Vehicles with earlier choices (larger utility) and later choices (smaller utility), it works well with the Breadth Linear Auction method due to its ability to maximize total utility, therefore avoiding large and small utility increases that do not increase the overall utility. In order to determine which assignment should be eliminated, the affect all assignments have on the system must be determined. In order to do this, the affect assignments in earlier cycles have on assignments in later cycles must be estimated (although an assignment may appear to be a good one during the cycle that it is made, the same assignment may result in poor future assignment choices). In order to consider this, the cut choice is made by comparing the added utility of the current assignment choice, and all future assignment choices by a Launch Vehicle, to the overall added utility of the current cycle; this value is them compared to all future cycles for all Launch Vehicles. This effectively gives the percentage of added utility for each branch of the tree. By comparing these numbers it is possible to determine which branch of the tree leads to the least amount of added utility.

By choosing this branch to be cut, it logically will eliminate poor decisions and lead to better choices.

4.6.2.1 Cut Choice Example

This example involves two Launch Vehicles, each with four Entry Vehicles. A network flow diagram of results from a solution method is shown in Figure 4-11. Here, the added utility of each assignment is listed in black next to the assignment path. The corresponding Cut choice matrix is shown below the figure in equation 16, where each column of the matrix represents a round, while each row represents a Launch Vehicle. In order to determine the percentage of added value, two sets of calculations must be completed. The first set of calculations is done to determine the amount of added utility

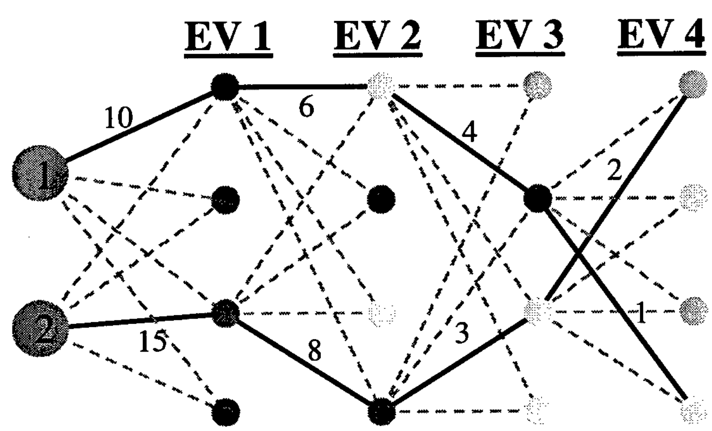


Figure 4-11. Network Flow with added Utility values shown

$$\begin{bmatrix} 10 & 6 & 4 & 1 \\ 15 & 8 & 3 & 2 \end{bmatrix} \tag{16}$$

from each assignment choice. To calculate this value, first the amount of added utility per round is determined and shown in equation 17. This is calculated by adding the

$$\begin{bmatrix}10+15 & 6+8 & 4+3 & 1+2\end{bmatrix} = \begin{bmatrix}25 & 14 & 7 & 3\end{bmatrix} \tag{17}$$

values in the columns of the matrix in equation 16 to determine the total value per column. In order to determine the added utility of each path, these values must be added, so that the first column equals the value of the first column plus all other values; the value in the last column will remain the same. The result for this is shown in the matrix in equation 18. The second set of calculations involve a similar calculation to that in

$$\begin{bmatrix}25+14+7+3 & 14+7+3 & 7+3 & 3\end{bmatrix} = \begin{bmatrix}49 & 24 & 10 & 3\end{bmatrix} \tag{18}$$

equation 18, where the values in each column are added to those in all subsequent columns to determine the total value of each path. The results for this calculation are seen in the matrix in equation 19. The final set of calculations requires a row-wise

$$\begin{bmatrix}10+6+4+1 & 6+4+1 & 4+1 & 1 \\ 15+8+3+2 & 8+3+2 & 3+2 & 2\end{bmatrix} = \begin{bmatrix}21 & 11 & 5 & 1 \\ 28 & 13 & 5 & 2\end{bmatrix} \tag{19}$$

division of the matrices in equation 18 and equation 19. This results in the matrix in equation 20, showing the percentage of added utility of each path for each assignment for

$$\begin{bmatrix}21/49 & 11/24 & 5/10 & 1/3 \\ 28/49 & 13/24 & 5/10 & 2/3\end{bmatrix} = \begin{bmatrix}.428 & .458 & .5 & .333 \\ .571 & .541 & .5 & .666\end{bmatrix} \tag{20}$$

each cycle. By choosing the lowest value, 0.333, as the assignment to be eliminated, the 4th assignment (assignment for 4th EV) for the first Launch Vehicle will be eliminated.

4.6.2.2 Cut Choice Knob

In the example in Section 4.6.2.1, the final choice resulted in the elimination of the assignment made by the first Launch Vehicle for its fourth Entry Vehicle. Although this

may result in a higher objective function value, the incremental increase is relatively small, since the sum of added utilities during the 4th round is 3 (approx 6% of total utility). In order to reach larger incremental increases it is best to eliminate poor decisions in earlier rounds. Although this method will result in some choices in earlier rounds, due to the averaging of values used in the development of the Cut Choice matrix, the majority of matrices will result in late round cuts. Figure 4-12 shows a plot of the

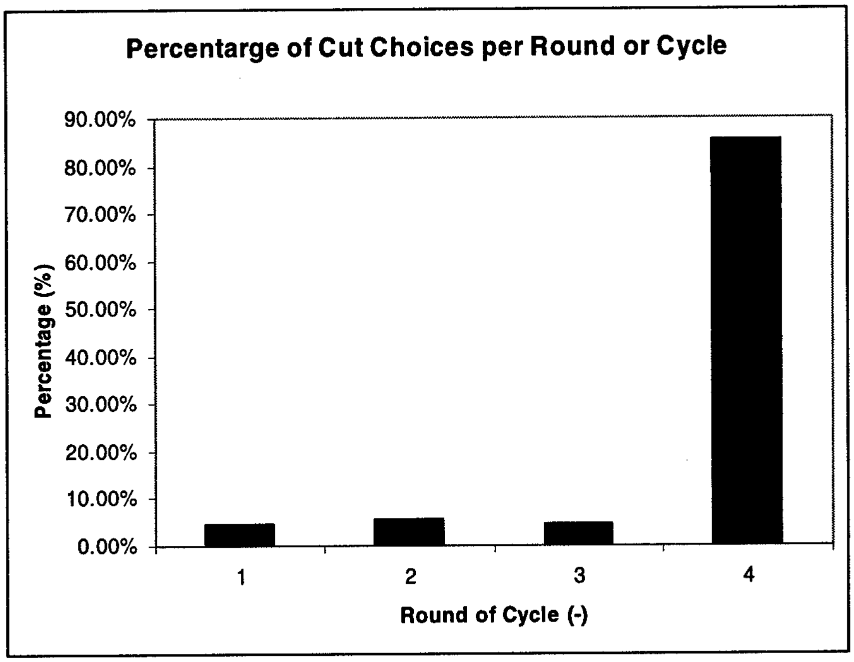


Figure 4-12. Percentage of Cut Choice per Round without Cut Knob

percentage of cut choices made per round using the directed search method for a ten Launch Vehicle, twenty target problem. Over 85% of all cut selections were made in the fourth round or cycle. Although this data only comes from one problem, this data is indicative of all problems. These results are due to the affect of averaging used in creating the cut choice matrix in equation 20. The values in the first through third column in equation 20 are an average of other values in later columns, helping to balance out any poor values, while those in the last column are not averaged. Because of this,

there is a high probability that the lowest value in the final column will be chosen as the assignment to be eliminated by the cut choice method.

In order to encourage more cut selections for earlier round assignments, a cut choice knob vector is used to change the cut choice matrix. This knob vector is calculated using a knob value, starting with a value of one in the right most column, and decreasing value in each subsequent column by the knob value. An example of a knob vector with a knob value of 0.05 is given in equation 21. Applying this knob vector to the cut choice matrix in equation 21 results in the matrix in equation 22. Although this matrix still results in

$$\begin{bmatrix} 0.9-0.05 & 0.95-0.05 & 1-0.05 & 1 \end{bmatrix} = \begin{bmatrix} 0.85 & 0.9 & 0.95 & 1 \end{bmatrix} \tag{21}$$

$$\begin{bmatrix} .428/0.85 & .458/0.9 & .5/0.95 & .333/1 \\ .571/0.85 & .541/0.9 & .5/0.95 & .666/1 \end{bmatrix} = \begin{bmatrix} .3638 & .4122 & .475 & .333 \\ .4854 & .4869 & .475 & .666 \end{bmatrix} \tag{22}$$

the same cut choice as that in Section 4.6.2.1, the results are much closer for other assignment choices, specifically the first Launch Vehicle's choice for its first Entry Vehicle, which is only 0.0305 above the lowest value as opposed to 0.095 in the earlier results. Running the directed search process again with a knob vector results in different percentage of round cuts; a plot of this result is displayed in Figure 4-13. By using a cut knob value of 0.05, the distribution of cuts was significantly moved towards the earlier rounds. Although there are still more cuts occurring in the later rounds, the percentage of cuts made in the second and third cycle are increased.

This knob can be used to affect which round a cut will likely occur in. Low knob values will result in later round cuts, while higher knob numbers will result in earlier round cuts.

Although neither type of cut is guaranteed to improve the solution, the ability to affect the round a cut is made in is an effective way of searching more possible solutions.

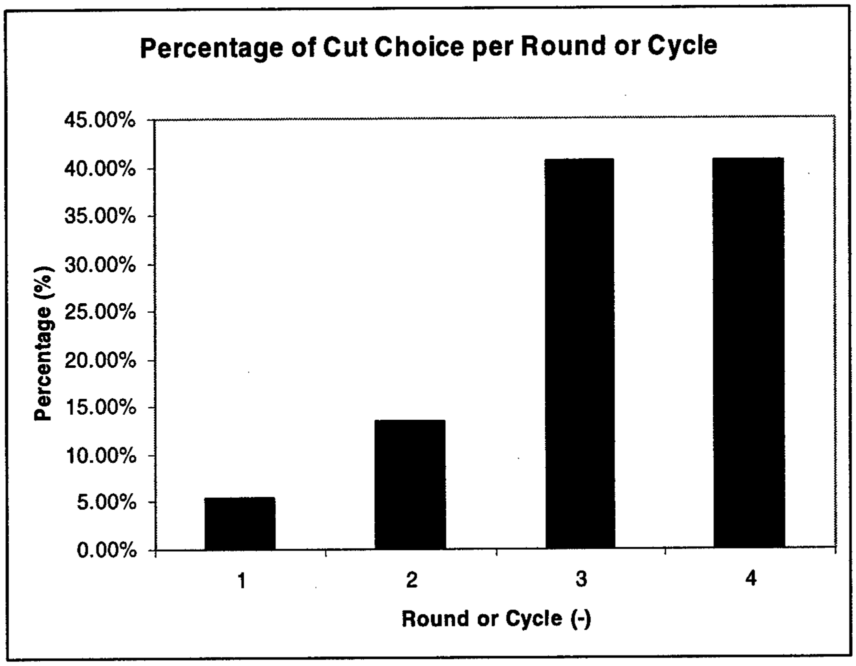


Figure 4-13. Percentage of Cut Choice per Round with Cut Knob of 0.05

4.6.3 Network Flow Representation of Directed Search

To get a better understanding of how the directed search method works it is useful to

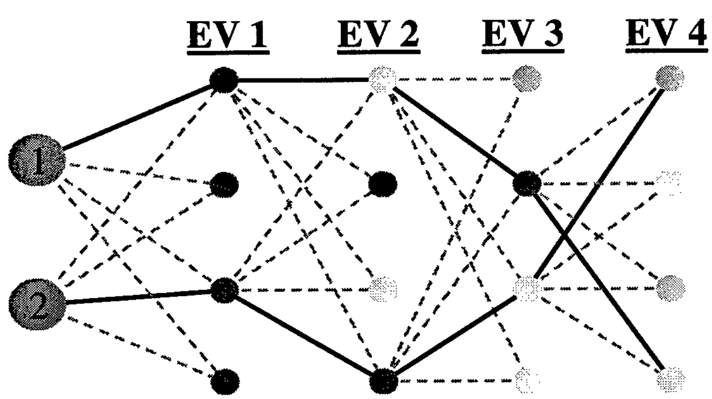


Figure 4-14. Network Flow Result of Breadth Linear Auction

show a graphical representation of the method. Figure 4-14 shows the results of a four target, two Launch Vehicle problem from one of the solution methods. Using these results as a starting point for the directed search method, a decision is made to eliminate the assignment for the first Launch Vehicle's second Entry Vehicle. This is represented in Figure 4-15 by a double red line showing a path that cannot be explored. The decision

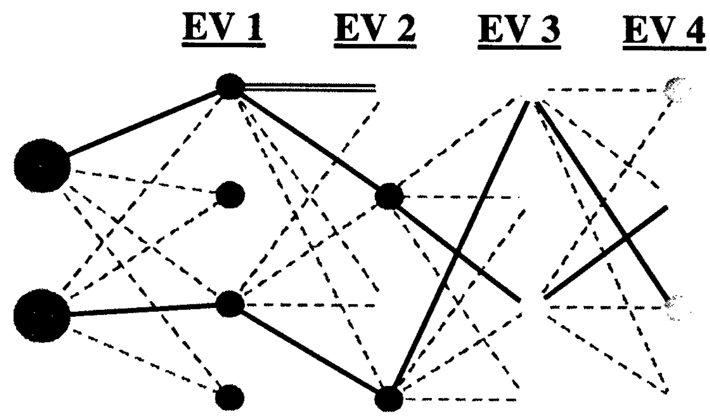


Figure 4-15. Network Flow showing result following Directed Search cut

to eliminate this assignment path does not affect assignments made before the eliminated assignment, but affects the assignments made after the eliminated assignment. Therefore, the assignment of both Launch Vehicles' first Entry Vehicle remains the same, but subsequent assignments change. In a situation where another assignment is eliminated in a further part of a branch, this elimination decision will remain because the first decision

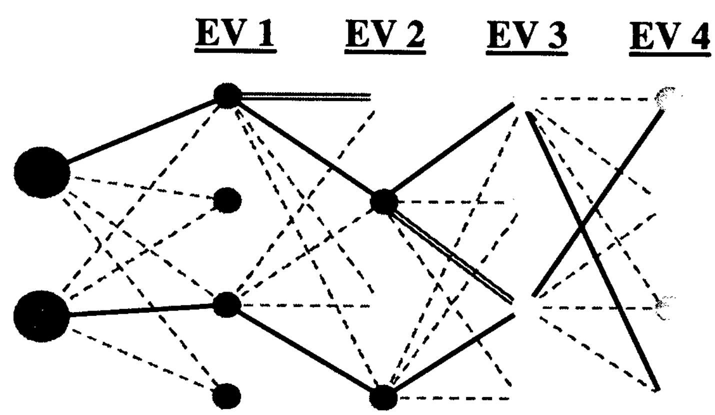


Figure 4-16. Network Flow showing result following 2nd Directed Search cut

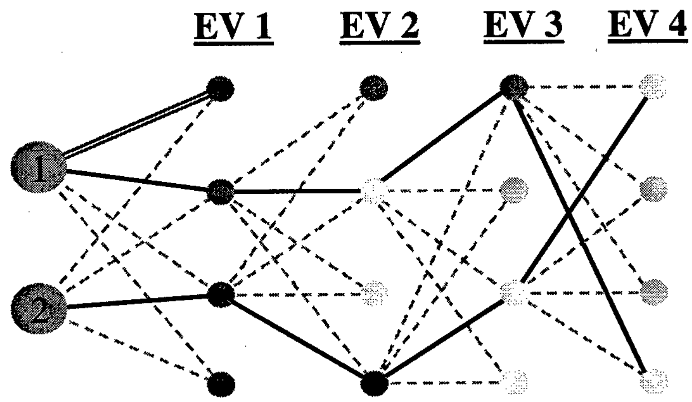


Figure 4-17. Network Flow showing result of Directed Search cut on early round

was not part of the branch cut by the second decision. This can be seen in Figure 4-16, where the assignment for the first Launch Vehicle's third Entry Vehicle is cut. Because the cut choice in the earlier round did not fall under the path of assignments made after the new eliminated assignment, it still appears in the figure and still affects assignment decisions. However, if the assignment for the first Launch Vehicle's first Entry Vehicle is eliminated such as in Figure 4-17, then all subsequent cuts are eliminated as well because they were in the branch of the path that was eliminated. This process will eventually lead to a point where enough assignment choices are eliminated that a viable solution does not exist; hence creating a situation where a linear auction cannot convergence to a solution.

4.6.4 Directed Search Convergence

Because the Directed Search method does not estimate the value of paths being eliminated or those remaining, each decision to cut a path does not guarantee an increase in the Objective Function value. Rather, the elimination of paths attempts to eliminate

paths that appear to be poor choices with respect to the value of other path selections based on the concept that the added utility from each path should approximately be the same. Therefore, the majority of cut choices will not result in an improvement in the solution, but as the process runs over time, solutions will be determined that are an improvement over the initial solution. An example of these results can be seen in Figure 4-18. These results are for a fifty target, twenty five Launch Vehicle problem, with a Cut Choice knob of 0.01. Each black circle represents the objective function value at the end of a directed search round. The blue line passes through all values that improve upon the best solution. This plot effectively shows how the majority of runs did not improve on the objective function value, but through numerous iterations, decisions paths were eliminated that led to improved results. This process can continue for a long time, but does not necessarily converge to the optimal solution. Because there is no way to know if the optimal solution has been reached, because there is no viable, accurate, method to

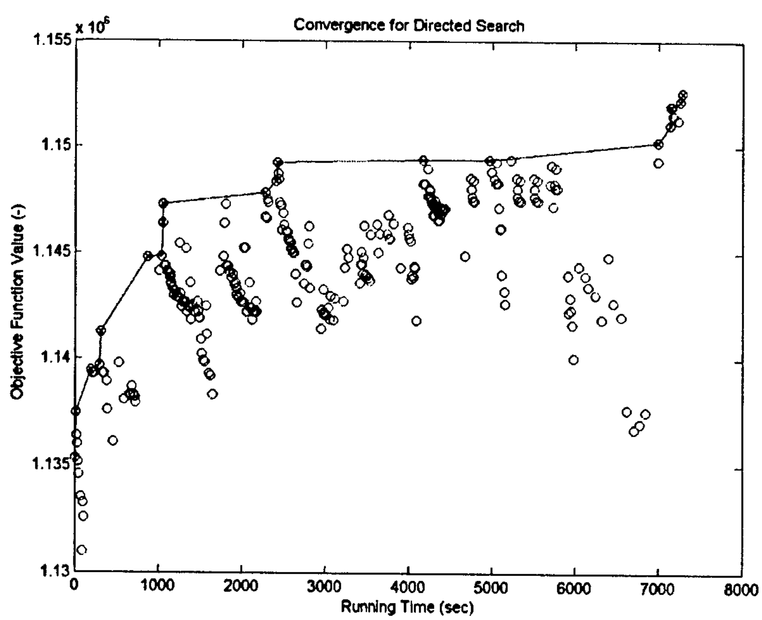


Figure 4-18. Directed Search Improvement

estimate the optimal solution, the process will not end if the optimal solution is met. Rather, the process will continue until either the user chooses to terminate the process, or the directed search method eliminates enough assignment paths that a viable solution no longer exists. If a viable solution no longer exists, the Linear Auction will not be able to converge due to its convergence requirement of a valid assignment for all Launch Vehicles (bidder). Although this ends the directed search, other directed searches could be run starting with the best results from the previous run, using a different cut choice knob, to possibly reach a better final solution.

4.6.5 Viable Solution Methods

Although the Directed Search method of eliminating poor assignments can be used with any of the Breadth based assignment methods, some solution methods are more or less effective than others. The Directed Search method is designed around the Breadth Linear Auction due to its ability to maximize the total increase in the objective function during each of its four cycles. In order to maximize the function, the process avoids high value paths for Launch Vehicles that force other Launch Vehicles into low value paths, resulting in solutions where each Launch Vehicle adds approximately the same amount of utility to the solution. Similarly, when an assignment is eliminated or cut by the directed search, the Breadth Linear Auction function is able to maximize the value for the current cycle and subsequent cycles, resulting in a higher probability of improvement in the solution. Although the Directed Search is developed with the intent of using the Breadth Linear Auction as the third and fourth steps of the Directed Search methodology (see Section 4.6.1), other methods can also be implemented. The Breadth Greedy method can

be implemented, but with much less improvement in performance. This decrease in performance is due to a difference in added utility from each Launch Vehicle. While the Breadth Linear Auction functions to balance the added utility of each Launch Vehicle so that each LV adds approximately the same amount to the Objective Function, all Greedy methods result in Launch Vehicles with earlier assignments with higher added utility, while those with later assignments with lower added utility. Because of this, using the Breadth Greedy method for assignments will result in the Directed Search method eliminating decisions for later Launch Vehicles, while eliminating very few assignments for Launch Vehicles that make decisions earlier. Figure 4-19 displays a plot showing the

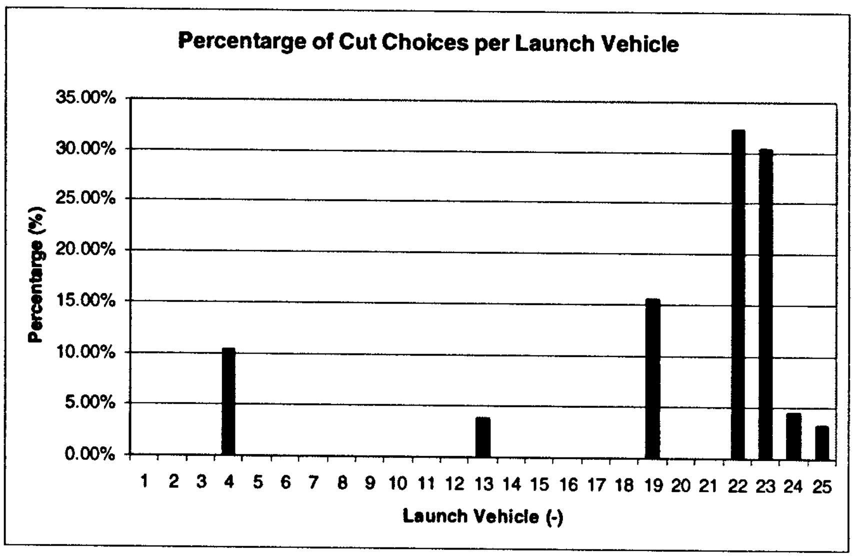


Figure 4-19. Percentage of Cut Choice per Launch Vehicle using Breadth Greedy

percentage of cuts chosen for each Launch Vehicle for a Directed Search using the Breadth Greedy method. These results are for a fifty target, twenty-five Launch Vehicle problem. Although some cut decisions were made for Launch Vehicles with early assignments (fourth and thirteenth Launch Vehicle), over 85% of cut choices occurred within the last seven Launch Vehicles. While cutting these choices will improve the

assignment solution, if an assignment of another Launch Vehicle is limiting the total solution it will not be cut due to the disparity in utility values between Launch Vehicles. The improvement from this method is compared to that of the Directed Search method using the Breadth Linear Auction method for assignment. Figure 4-20 displays the

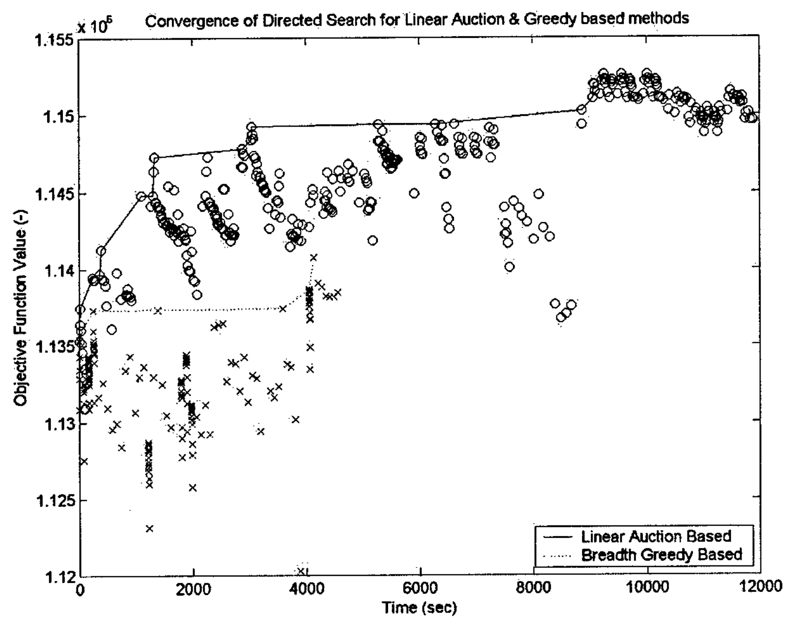


Figure 4-20. Convergence graphs for Directed Search w/Auction vs. Greedy

improvement of each method over time. The black solid line represents the best current solution for the Breadth Linear Auction driven Directed Search, while the dotted blue line represents the best current solution for the Breadth Greedy driven Directed Search. The black circles are results from the cycles of the Breadth Linear Auction drive method that did not improve on the best solution, while the blue Xs represent results that do not improve the solution for the Breadth Greedy based method. Although the Breadth Greedy driven Directed Search improves on the initial assignment, it does so at a much slower pace than the Breadth Linear Auction driven Directed Search. The Breadth Linear

Auction method was able to reach a better solution than the Breadth Greedy method, with a solution time of less than ten minutes compared to over an hour.

4.7 Randomly Perturbed Weight Breadth Linear Auction

This method uses the Breadth Linear Auction, with its strengths and weaknesses, to attempt to improve on the assignment solution by running numerous Breadth Linear Auctions with randomly perturbed target weights. Each Breadth Linear Auction is run in an attempt to maximize the results for a set of randomly perturbed target weights, with final results calculated using the original weights. Although this method does not guarantee an improvement over a regular Breadth Linear Auction, improvements will occur if a sufficient number of tests are completed. Each test using random weights takes the same amount of time to complete as a Breadth Linear Auction. For this reason, it will take a considerably long time to find improved solutions to large problems.

4.7.1 Randomly Perturbed Weights

In order to best search the space close to the optimal solution, the random variables were created using equation 23 below, where the function rand creates a random variable following a Gaussian distribution with a mean of W(k) and a standard deviation of σ.

$$W_{rand} = \left| rand(W(k), \sigma) \right| \tag{23}$$

The final random weighted value is the absolute value of this function to ensure positive weights (negative weights would cause targets to be avoided). This equation effectively creates random perturbations on target weight weights, with more occurrences near the

original weight, with some weights as large or small as the largest and smallest original weights. During all example problems, weights were chosen as an integer value from one to five. Because of this, and to ensure that each random weight can fall anywhere within the spectrum, a three standard deviation (3σ) of 5 is used. In order to get a better understanding of these random weights, Figure 4-21 shows a histogram of randomly

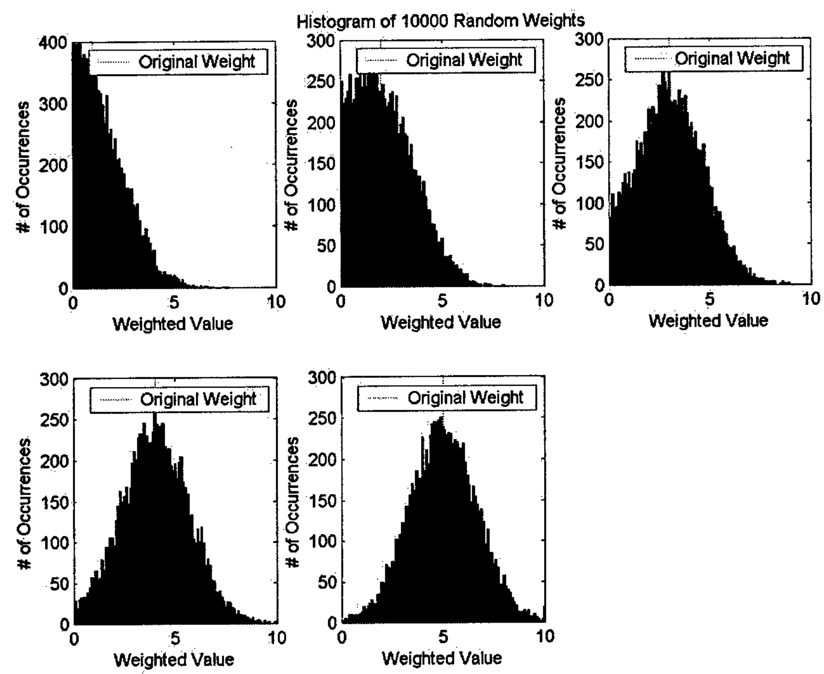


Figure 4-21. Distributions of Randomly Perturbed Weights for integers from 1 to 5

perturbed weights created for starting weights of one through five. As the plots show, random weights for higher values such as 4 and 5 closely follow a Gaussian distribution because only a small percentage of their values from the rand function are negative. However, for lower values, specifically starting weights of 1 and 2, the plots are not very Gaussian. Although these don't follow a Gaussian distribution, they do affectively disturb the weights in a desired manner; for all plots, the majority of values occur near

the original value, with maximum and minimum values reaching both the maximum and minimum starting weights.

4.7.2 Random Weights Example Results

A histogram is used to analyze the results using randomly perturbed weights due to the number of tests necessary to reach an improved solution. Figure 4-22 displays a

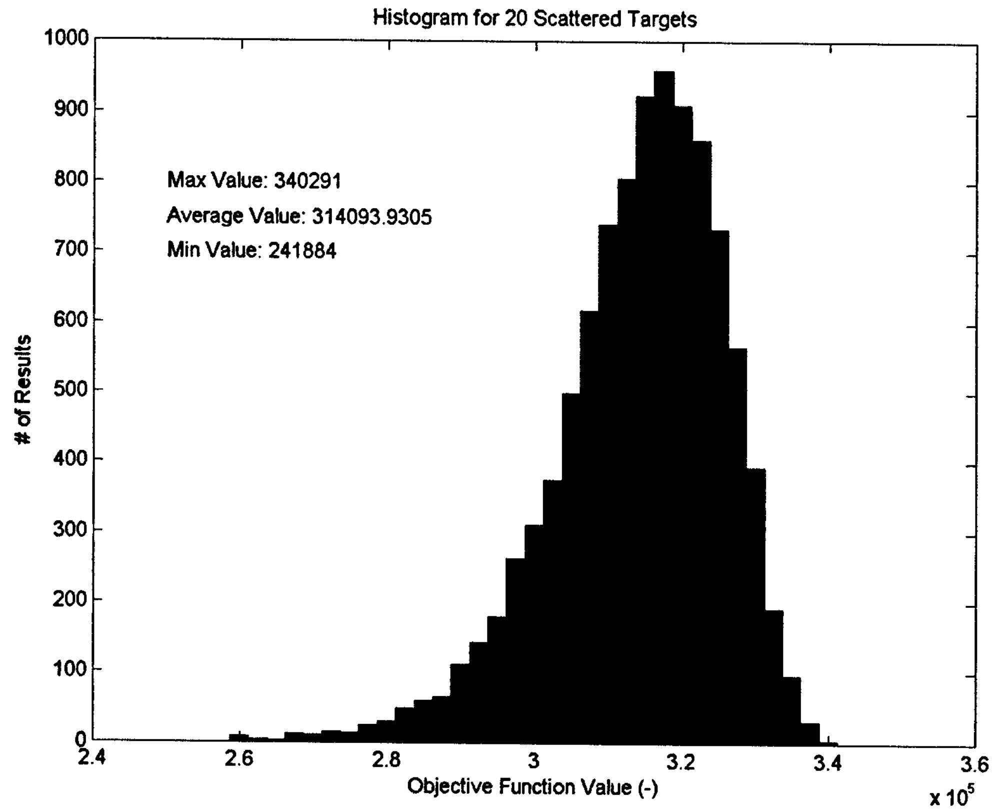


Figure 4-22. Example Histogram of Randomly Perturbed Weight Linear Auction

histogram for a ten target, twenty Launch Vehicle problem. The histogram is for 10,000 tests, with each column representing the number of tests that resulted in a solution with an Objective Function within its bounds. The histogram can be used to estimate the

number of runs of the Randomly Perturbed Weight Breadth Linear Auction necessary to result in an improvement over a regular Breadth Linear Auction.

4.8 Comparison Values

Although solutions from the various implementations of the Breadth and Depth based methods can be compared to each other, these comparisons can be put in context when compared to values such as the minimum possible value, or the optimum value. The easiest way to reach an optimal value is through the use of a brute force search. However, for problems of size (anything with more than two Launch Vehicles), it is computationally impossible to reach a Brute Force solution. Therefore, in order to develop comparison values, a number of random assignments were determined for each problem type. The distribution of these assignments was used to determine a maximum, minimum, and average value of the distribution, as well as the standard deviation of the distribution. These values will be used when comparing values of assignment solutions from different solution methods.

All results with respect to time were recorded on a Pentium 4, 2.19 GHz processor with 1.0 Gb of RAM while running uncompiled Matlab© code. The relative difference between results can be used to compare the average time per calculation.

4.8.1 Brute Force Method

The Brute Force search technique will result in an optimal solution for all optimization problems. However, due to the numerical complexity of most problems, the Brute Force method is not a viable option for reaching solutions. This is true for the nonlinear assignment problem at the focus of this research. Although it is feasible to solve very small assignment problems, it is not possible to solve problems of significant size. The following sections discuss the solution for a small assignment problem, as well as the computational cost of solving larger problems.

4.8.1.1 Brute Force results for a small assignment problem

An assignment problem with four targets and two Launch Vehicles was solved using both a brute force method and all variations of the both the Breadth and Depth based Methods. A comparison between these methods is shown in Table 4-2. The final objective

Table 4-2. Comparing results of Breadth and Depth Methods to Brute Force

	Objective Function	Solution Time (s)	% Imp over Brute Force	x Times Longer
Brute Force	139001	498		
Breadth Greedy	139001	1.391	0.00%	358.015816
Breadth Linear Auction	137345	1.719	1.21%	289.703316
Depth Greedy	139001	1.437	0.00%	346.555324
Hybrid	137345	2.547	1.21%	195.524146

function value, the number of standard deviations each solution is from the mean solution, and the solution times are listed. A comparison between these solutions and the Brute Force solution are given. Both Greedy methods reached the same optimal solution as the Brute Force method, and both Auction methods reached solutions with slightly lower values (same assignments, but different inclination and release time choices). All

of these solutions were reached considerably faster than the brute force method, with an average savings of 496 seconds (297x).

4.8.1.2 Computational Effort for Brute Force method

Although the computational effort for the simple problem in Section 4.8.1.1 is acceptable, the computational time for larger problems is not acceptable. This is a result of problem expansion due to the increase in both number of targets and number of assignments to be made. For the four target, two Launch Vehicle problem, a total of 2,016,270 different solutions were searched. For a six target, three Launch Vehicle problem, the total number of solutions to search increases to 3.372E10, an increase of over 16,000 time in size. An even larger increase occurs for an eight target, four Launch Vehicle problem,

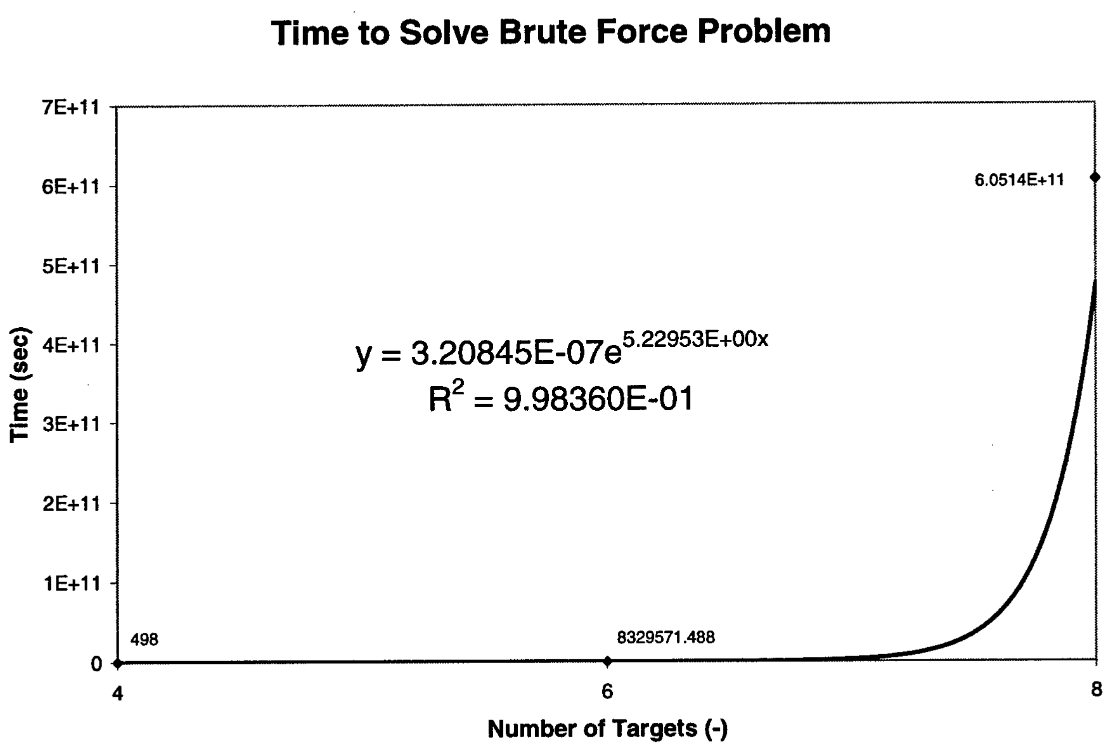


Figure 4-23. Solution time for three Brute Force problems

with the total number of solutions increasing to 2.450E15; an over 72,000 time increase in size from the six target, three Launch Vehicle problem. Assuming it takes the same amount of time to reach each solution (the time will actually increase for larger problems), the six target problem will take over 96 days to solve, while the eight target problem will take nearly twenty millennia to solve. A plot of these times is seen in Figure 4-23, with an exponential trend line. This equation can be used to calculate solution times for larger problems. However, it is evident from smaller problems that the computational effort to reach brute force solutions is excessive.

4.8.2 Random Assignments

In order to develop a baseline to compare results, thousands of random assignments were made to develop a histogram of results. This histogram makes it possible to estimate the average Objective Function value of the average assignment, as well as giving an estimate of the lowest and highest possible values of the objective function and a standard deviation of possible results. Each random assignment is chosen by randomly (uniform random) selecting a launch inclination and release time. Once this selection is made, a random (uniform random) selection of targets is made for those with positive $P(S_f \mid S_{EV} \cap S_{LV})$ values for the inclination and release time. This process is completed for every Launch Vehicle, and the Objective function is calculated at the end of the process. The histogram in Figure 4-24 was compiled of 100,000 random assignments for the four target, two Launch Vehicle problem discussed and solved using the brute force method in Section 4.8.1.1. As stated on the plot, the maximum value is 136,601, while

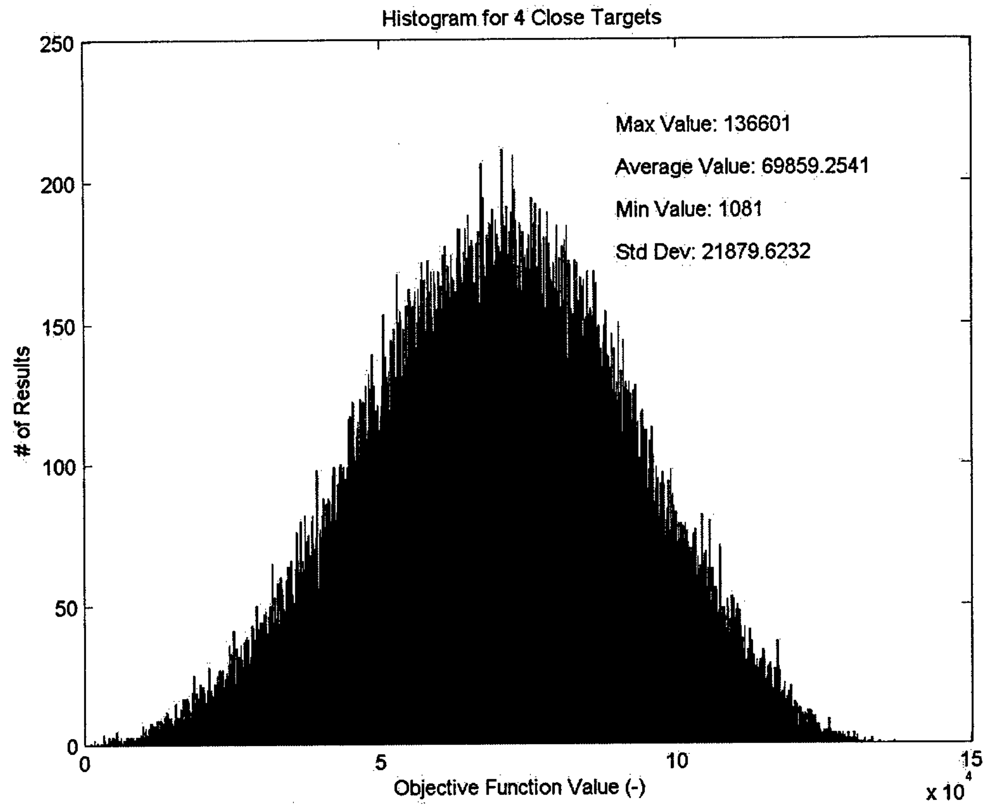


Figure 4-24. Histogram of Random Assignments for four Target, two LV problem

the minimum value is 1,081. Although the maximum value is close to the optimal value of 139,001, it still did not result in any assignments that equaled the maximum value. This leads to reasoning that for larger problems, the maximum random assignment will not result in an optimal value, and may in fact result in a larger difference between the estimated optimal value and the actual optimal value.

5 Results

In order to examine the effectiveness of the different assignment methods, a total of six problems were developed and solved using each method. Two of these problems are used to show the weaknesses and strengths of both the Breadth and Depth based methods, while four of these problems use variations in numbers of targets and how closely the targets are grouped to show the effectiveness of the different solution methods. For the first two problems, the weighting of targets is varied to show situations in which both the Breadth and Depth based methods have difficulty finding solutions. For the final four problems, results are compared between all Breadth and Depth based methods, with comparisons made based on the maximum, minimum, average value, and standard deviation of numerous random assignments for each problem. Convergence results showing solution improvement using a Breadth Linear Auction driven Directed Search, starting from various initial assignments, are used to compare the effectiveness of the Directed method. Using histograms, an estimate of the amount of time necessary to reach an improved solution using the Randomly Perturbed Weight Breadth Linear Auction is shown for each problem

5.1 Comparison between Solution from Different Methods

Because both breadth and depth based methods have different strengths and weaknesses, different problem types can be used to compare their performance. Two different target locations are used for a six target, three LV problem, each with a grouping of four highly weighted (5) targets, with two separate targets. Solutions are compared between Breadth

and Depth based methods for integer target weights for the separated targets from one to five. Because each of the problems is relatively small, both of the Breadth methods result in the same solution, and both the Depth Greedy and Hybrid methods result in the same solution. Therefore, each of these problems will compare one solution for both the breadth and depth methods.

5.1.1 Breadth Advantage vs. Depth Advantage target placement

In order to examine problems that enhance the differences between the Breadth and Depth based methods, two simple problems were developed that result in large differences between the two methods. Each problem involves six targets, with four grouped targets and two separated targets. In the problem setup that favors the Breadth based method, the two separated targets are placed far enough away from other targets that their selection will result in a release time and inclination selection that forces a

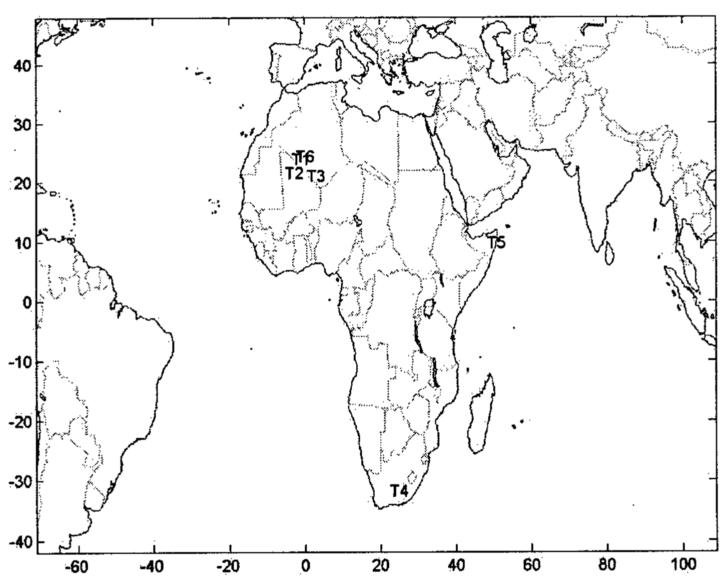


Figure 5-1. Target Location for Breadth Advantage problem

Launch Vehicle to assign all of its Entry Vehicles to the target. A picture of the target locations for the Breadth advantage problem is shown in Figure 5-1. In the problem

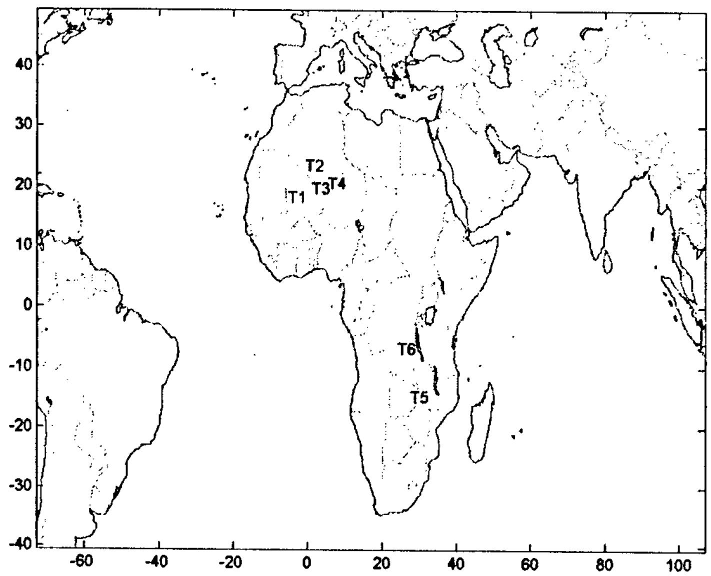


Figure 5-2. Target Location for Depth Advantage problem

setup that favors the Depth based method, the two separated targets are grouped closer together so that a single release time and inclination can have a positive $P(S_f \mid S_{EV} \cap S_{LV})$ value for both targets. However, these targets are separated enough from the group of four targets that it is impossible for a Launch Vehicle to have EVs that target both the separated targets and the four grouped targets. Figure 5-2 shows the target locations for the Depth advantage problem.

5.1.2 Breadth Advantage Results

Due to the target location in the Breadth advantage problem, if a Launch Vehicle selects one of the separated targets it will have to assign all of its Entry Vehicles to that target.

When the separated targets are weighted lower than the grouped targets (5), the Breadth method will result in all three Launch Vehicles assigning targets in the grouping to their first Entry Vehicles, resulting in all Launch Vehicles assigning all Entry Vehicles to those within the grouping. The breadth method, however, will assign some Launch Vehicles' first Entry Vehicles to the separated targets because the assignments will be made after other Launch Vehicles have assigned the grouped targets, resulting in a decrease in value for retargeting the group. If the separated targets have low weights, the value of assigning Launch Vehicles will be low. Figure 5-3 compares the objective function value

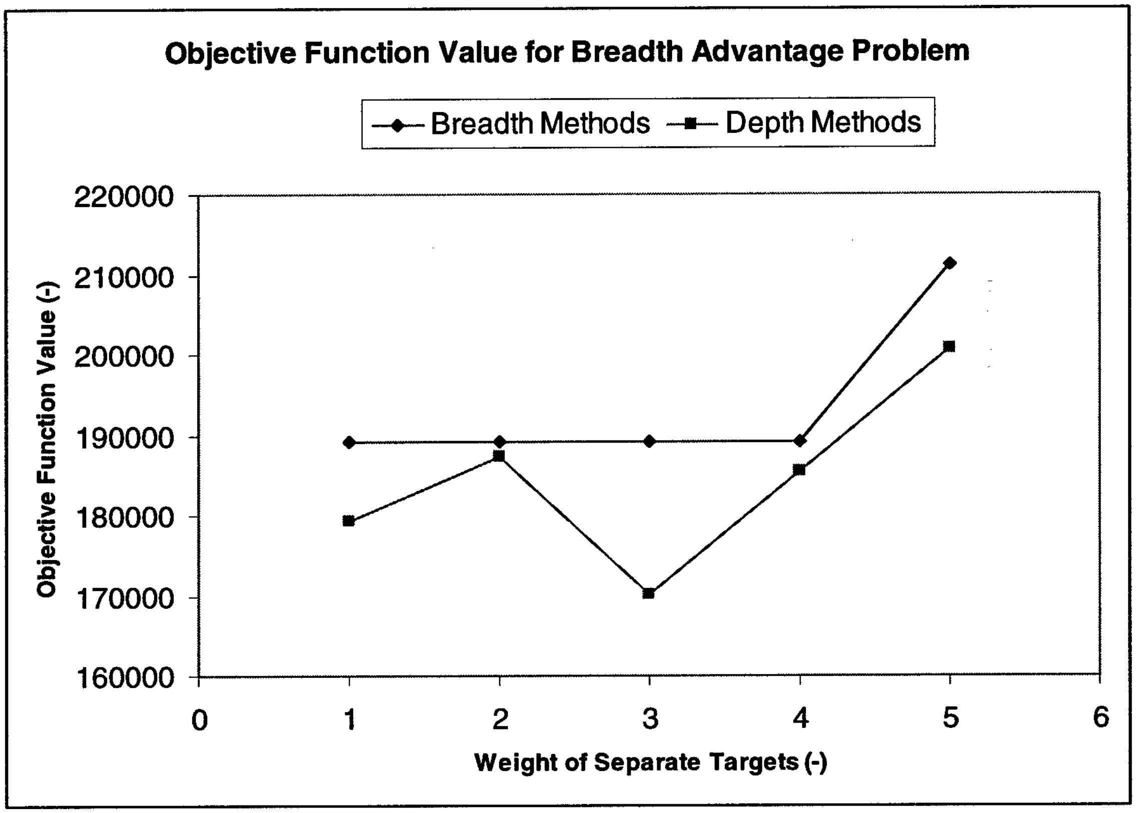


Figure 5-3. Comparison of Standard Deviation of Results of Breadth and Depth based methods to Breadth Advantage Problem

between Breadth and Depth based methods for the Breadth advantage problem. For the solutions with a target weight for the separate targets of one and two, the Depth Method

assigns two Launch Vehicles to the grouped targets and one to a separate target. For the solutions with a target weight for the separate targets of three to five, the Depth Method assigns one Launch Vehicle to the grouped targets, and one to each of the separated targets. This results in situations where the Depth Method's objective function value is close to that of the Breadth Method, and situations where they are separated in value. For the situation where the separate targets have a weight of one, the Depth method results in a lower solution because the EV assigned to the target cannot add much total value (one) to the objective function. When this value is raised to two, this Launch Vehicle can add more value (two) to the objective function, resulting in a solution that is closer to the solution from the Breadth Method. When the weight increases to three, the Depth method assigns a second Launch Vehicle to the separate targets. Although both of these now add more utility to the total Objective Function, the high weighted group of targets is now only targeted once, resulting in a decrease in the Objective Function value and a large disparity between results. This distance decreases when the separated targets increase in value again. When the separate targets have the same weight (five) as the grouped target, the Breadth method also assigns one Launch Vehicle to the separate targets. This results in an increase in the objective function value for the Breadth methods. Although the difference between results is relatively small, this size is dependant on the number of and weights of targets as well as Launch Vehicles. If more targets were included in the problem, a larger gap would occur. This shows that for certain problems, even though the Depth method attempts to make good first Entry Vehicle assignments, it can make bad assignments because it cannot predict available value from a release time and inclination.

5.1.3 Depth Advantage Results

The target setup for the depth advantage problem is similar to that of the breadth advantage problem; however the depth problem groups the two scattered targets. This allows Launch Vehicles that target the separated targets to split their Entry Vehicle assignments between the two targets. This increases the capability to reach higher objective function values by sending some Launch Vehicles to target the separated

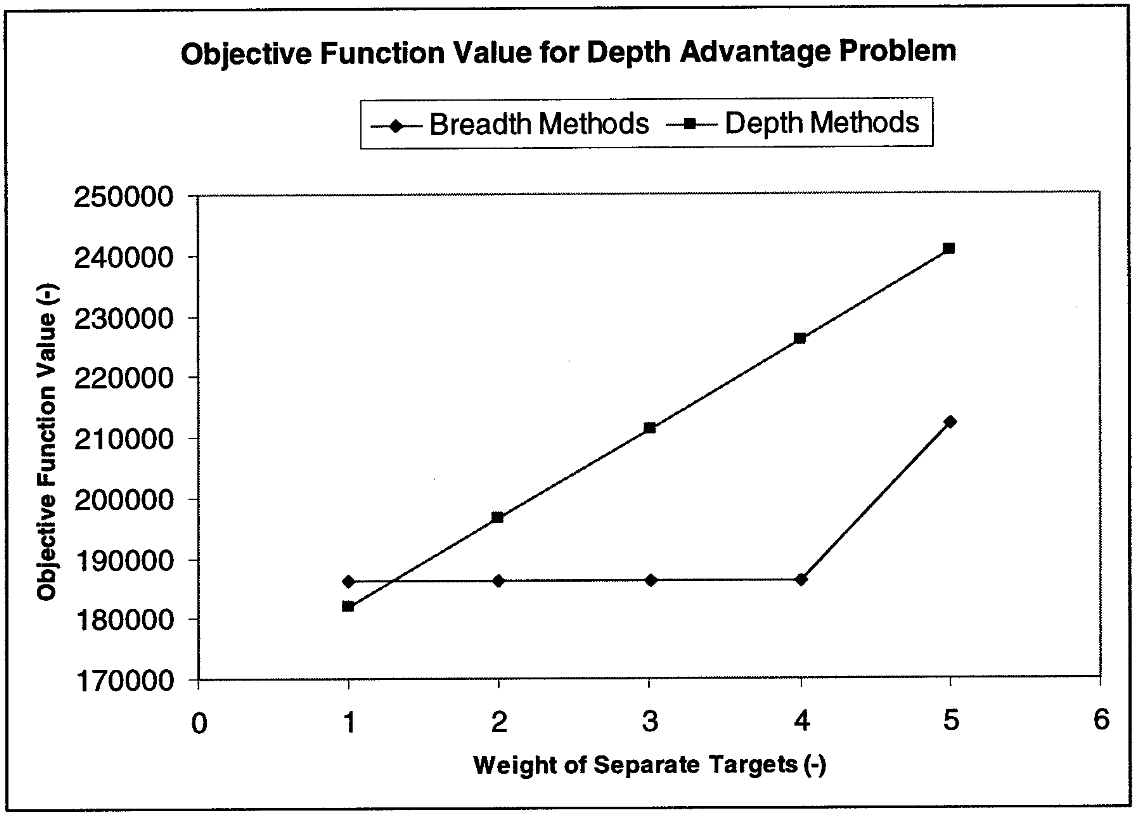


Figure 5-4. Comparison of Standard Deviation of Results of Breadth and Depth based methods to Depth Advantage Problem

targets. Figure 5-4 compares the results of the breadth method to the depth method for the depth advantage problem. Throughout weights of one to four, the breadth method assigns all Launch Vehicle's EVs to targets in the group, while the depth method assigns two Launch Vehicle's EVs to targets in the group and one Launch Vehicle to the separate

targets; when the separate targets have a weight of five, the Breadth Method assigns one EV to the scattered targets, and two to the grouped targets. When the separate targets have a low weight, the breadth method results in a slightly higher objective function value; however, as the weight increases a gap begins to appear between the objective function values, because the Breadth methods value does not increase as the weights of the separated targets increase. This result shows that grouping high value targets can cause Breadth based methods to make poor selections, resulting in all Launch Vehicles targeting locations with overlapping footprints. However, similar to the Breadth Advantage problem, the difference in value is relatively small. However, this could increase if more targets, more Launch Vehicles, or higher weights were used.

5.2 General Problems

Although Space Based Entry Vehicles will be used in various targeting situations, with different types of target grouping and numbers of targets, for academic purposes four problems will be posed with randomly weighted targets and target locations. These problems include two problems with twenty targets and two with fifty targets; one of each set of problems includes scattered targets and closely grouped targets. Since it is impossible to determine the optimal solution to these problems, numerous random assignments were made in order to determine a distribution of objective function values for assignments. This distribution yields a maximum, minimum, and average value as well as a standard deviation which will be used to compare the different solution methods for each of the four problems. Results are also shown for numerous Directed Search methods starting from assignments reached using one of the breadth or depth based

methods. Using histograms created from numerous Breadth Linear Auction with randomly perturbed weights, estimates are made for the amount of time necessary to reach an improved solution using randomly perturbed weights.

5.2.1 Problem Sizes

Many variables related to targets can affect the speed and quality of solutions developed from breadth and depth based methods. These variables include target location, weight, kill method (regular warhead or unitary penetrator), and the number of targets. The largest changes in solution quality include both number of targets and target locations. Because of this, four problems with variations of both variables are made. These problems include small numbers of targets (twenty) and larger numbers of targets (fifty), as well as scattered target location (throughout Africa) and closely grouped target location (box in upper corner of Africa). Both the weighting of targets and kill method are chosen randomly, with a uniform chance of resulting in a weight from one to five, and an equal chance of targets being destroyed by a regular warhead and unitary penetrator. Figure 5-5 shows the target location of the twenty scattered targets, with targets scattered throughout Africa. Figure 5-6 displays the location of the twenty closely grouped targets. All twenty are grouped within a square in the upper corner of Africa. The same types of groupings are used for the problems with larger number of targets, with considerably more targets in each grouping. Figure 5-7 displays the target locations for the fifty scattered target problem, with targets spread throughout the continent of Africa. The target location for the final problem setup is displayed in Figure 5-8, with fifty target

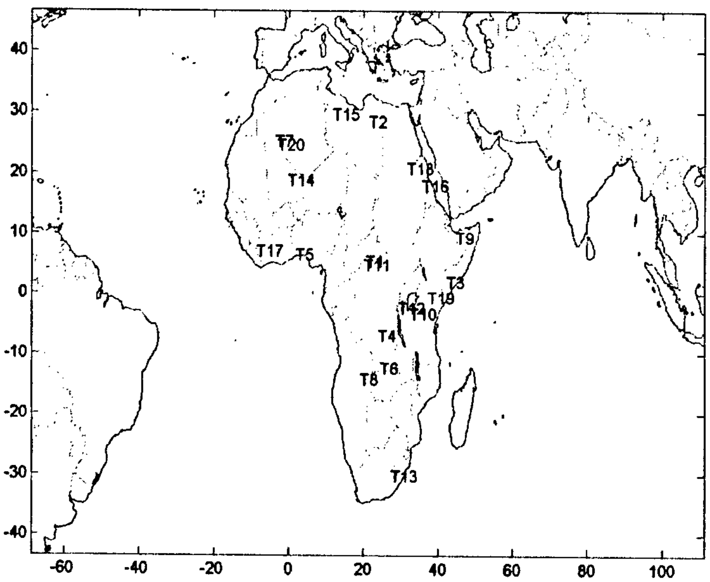


Figure 5-5. Target Location for 20 Scattered targets

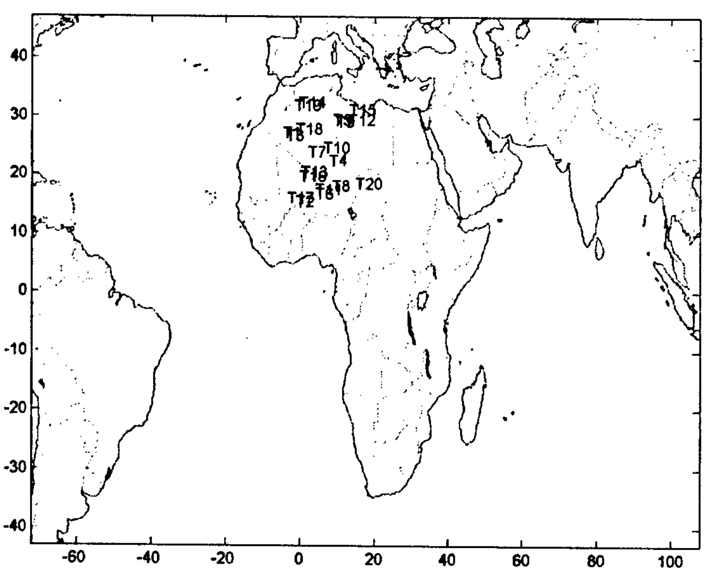


Figure 5-6. Target Location for 20 Close targets

spread randomly throughout the same square as the problem with twenty close targets. These four problem setups will be used to compare results between the various Breadth and Depth based solution methods as well as the Directed Search and Randomly

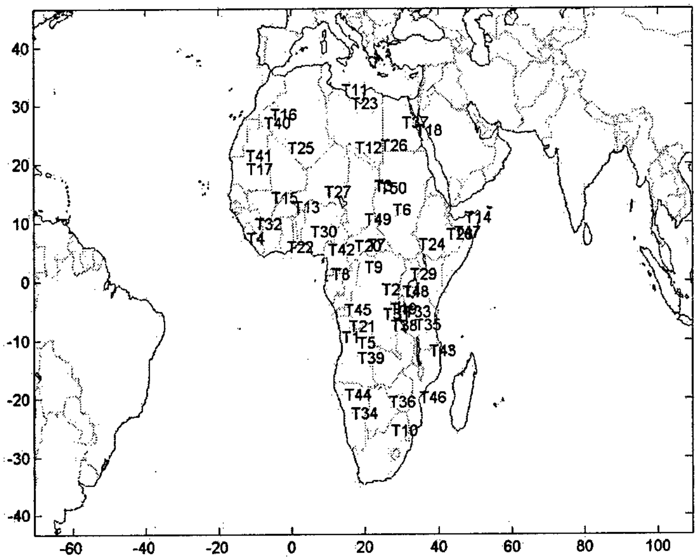


Figure 5-7. Target Location for 50 Scattered targets

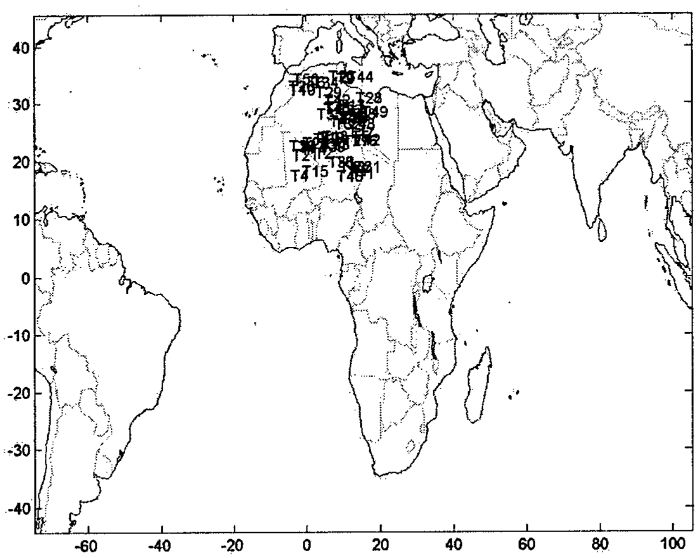


Figure 5-8. Target Location for 50 Close targets

Weighted Breadth Linear Auction methods. These results will be discussed in the subsequent sections.

5.2.2 Comparison Values

As discussed in Section 4.8.1, it is impossible to determine the optimal assignment using brute force methods. Therefore, in order to develop comparison values from which to compare solutions, a distribution of objective function values for random target assignments is used. The maximum, minimum, and average value, as well as the standard deviation of these distributions, is used in comparing solutions from the various target assignment methods (both Breadth and Depth based) as well as the Directed Search method. For all four problem types, a total of 100,000 random runs were completed. A histogram of the results for each run are plotted in order to give a graphical

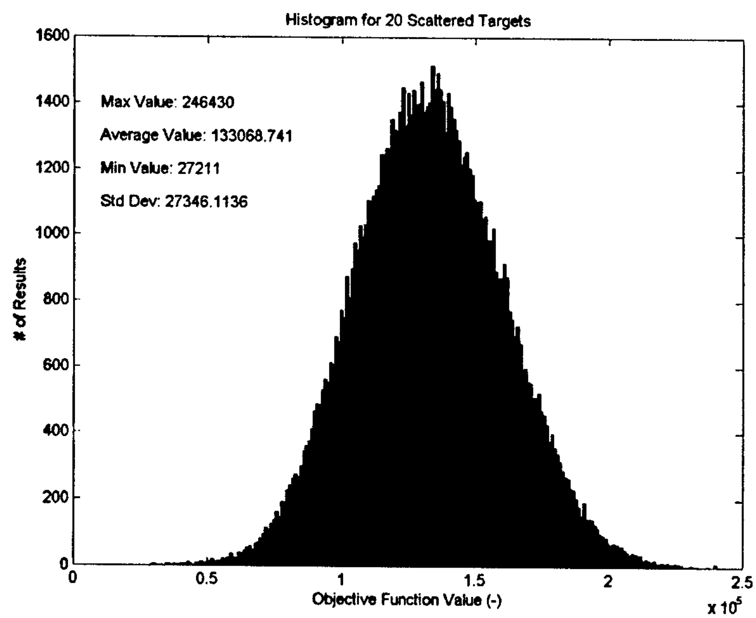


Figure 5-9. Histogram of Random Assignments for 20 Scattered targets

representation of the objective function distribution. Figure 5-9 displays the histogram for the twenty target problem, with the maximum, average, and minimum values, as well as the standard deviation of the results listed. Because the histograms for the other problems will be similar, there are not shown; rather, their results are tabulated in Table

5-1. These values will be used in subsequent section to compare the solutions obtained by the reference breadth and depth methods as well as the Breadth Linear Auction and Hybrid method.

Table 5-1. Reference Values for four Basic Problems

	Random Assignments			
	Min Value	Average Value	Max Value	Standard Deviation
20 Scattered Targets	27,211	133,069	246,430	27,346
20 Close Targets	19,050	128,786	283,119	34,516
50 Scattered Targets	164,180	384,649	637,516	58,011
50 Close Targets	186,499	410,146	658,826	55,908

5.2.3 Results from Breadth and Depth Based Methods

In order to compare results from the basic (Greedy) Breadth and Depth based assignment methods and those that include the use of a Linear Auction, comparisons are made between the objective function values for each method. In order to more accurately compare results, the number of standard deviations each result is from the mean value of the random assignment distribution (see Table 5-1) is listed, giving a reference of the solution methods. Table 5-2 lists the objective function results for the twenty scattered

Table 5-2. Objective Function results for 20 Scattered target problem

	Objective Function	Time (s)	% Imp Over Rand Max	Std Dev over Rand Mean
Breadth Based Method				
Linear Auction	337,828	24.81	41.69%	7.49
Greedy	337,115	19.42	41.37%	7.46
Depth Based Method				
Greedy	334,268	33.47	40.07%	7.36
Mixed Method				
Hybrid	333,810	53.36	39.86%	7.34
Random Max	246,430	566.17	0.00%	4.15

target problem. The lowest standard deviation (and objective function) is for the hybrid method, with an objective function value of 7.34 standard deviations greater than the

average value, and a 39% increase in the objective function value over the maximum value from the random assignments. This large standard deviation shows that both the Breadth and Depth based methods do a sufficient job of finding high value solutions, and that it'd be nearly impossible to find a similar solution from random assignments. The best solution comes from the Linear Auction breadth method, with an objective function value 7.49 standard deviations greater than the average value. This is a 0.03 improvement over the Breadth based greedy method due to the use of Linear Auction. Table 5-3 lists the objective function values for the twenty close target problem. For this

Table 5-3. Objective Function results for 20 Close target problem

	Objective Function	Time (s)	% Imp Over Rand Max	Std Dev over Rand Mean
	Breadth Based Method			
Linear Auction	423,510	25.62	53.16%	8.54
Greedy	422,421	17.20	52.75%	8.51
	Depth Based Method			
Greedy	417,944	29.61	51.06%	8.38
	Mixed Method			
Hybrid	429,076	47.50	55.27%	8.70
Random Max	283,119	589.57	0.00%	4.47

and all subsequent problems, the hybrid method results in the best assignment solution, with an objective function value 8.70 standard deviations greater than the mean value of the random assignment distribution. Again, the Linear Auction breadth method results in a 0.03 increase in the number of standard deviation from mean over the greedy Breadth based method, showing the improvement of optimizing the results of each round. Table 5-4 displays the results for the fifty scattered target problem, while Table 5-5 displays the results for the fifty closely grouped target problem. The best result for both problems is the hybrid method, with an objective function value of 13.20 standard deviations over the mean (SDM) for the scattered problem an objective function value of 14.09 SDM for the closely grouped problem. Both problems show a larger increase in value for the Linear

Table 5-4. Objective Function results for 50 Scattered target problem

	Objective Function	Time (s)	% Imp Over Rand Max	Std Dev over Rand Mean
Breadth Based Method				
Linear Auction	1,145,041	356.46	107.22%	13.11
Greedy	1,137,438	267.74	105.62%	12.98
Depth Based Method				
Greedy	1,146,088	771.13	107.44%	13.13
Mixed Method				
Hybrid	1,150,168	1055.60	108.31%	13.20
Random Max	637,516	1958.30	0.00%	4.36

Table 5-5. Objective Function results for 50 Close target problem

	Objective Function	Time (s)	% Imp Over Rand Max	Std Dev over Rand Mean
Breadth Based Method				
Linear Auction	1,195,217	480.58	113.56%	14.04
Greedy	1,178,538	394.83	110.03%	13.74
Depth Based Method				
Greedy	1,181,951	663.44	110.75%	13.80
Mixed Method				
Hybrid	1,197,963	1056.60	114.14%	14.09
Random Max	658,826	2090.30	0.00%	4.45

Auction over the greedy breadth method, with an increase of 0.13 SDM for the fifty scattered problem and 0.30 SDM for the fifty closely grouped problem.

These solutions show that there are numerous poor solutions for each of the four problems, with very few high objective function value solutions. However, these solutions can be found easily using any of the incremental solution methods (Breadth or Depth). However, the best solutions are found using the Linear Auction based methods (Linear Auction or Hybrid).

The Linear Auction and Greedy result in similar solutions for each problem, with the Linear Auction resulting in an average increase of 0.12 standard deviations from the mean, at an extra computational effort between 5 seconds and a minute and a half. The Depth Greedy method takes longer to reach a solution due to longer calculation times for

the assignment of each LV's first EV. The Hybrid method takes the longest to reach a solution, with solution times equal to the combination of both a Depth Greedy and ¾ of a Breadth Linear.

5.2.4 Directed Search Results

The directed search can be used to improve on the results from other solution methods. As discussed in Section 4.6 and subsequent sections, the Directed Search method uses the breadth linear auction to improve on results by determining poor assignments and eliminating this path through the network. A new assignment is determined by

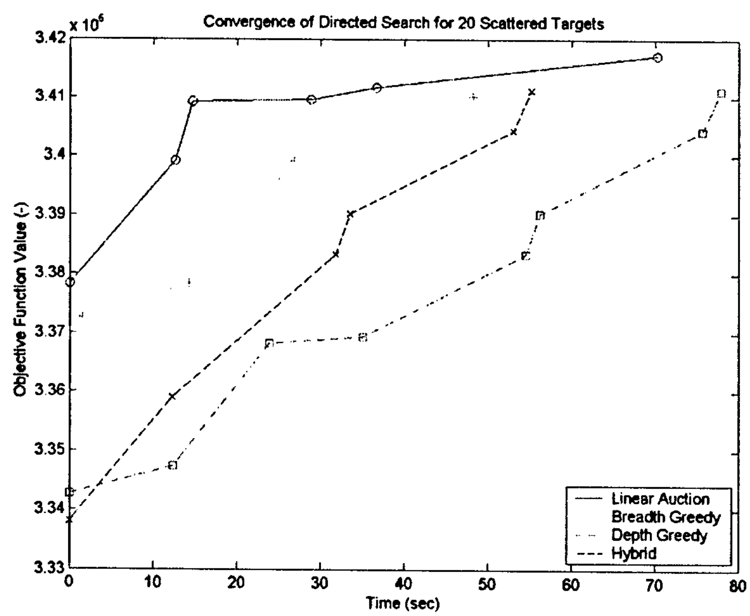


Figure 5-10. Convergence for Directed Search for 20 Scattered target problem from various Starting Assignments

recalculating the rest of the assignments. This process can be repeated, resulting in different improved solutions. Directed Searches were completed for each of the twenty target problems, starting with the assignments that resulted in the objective function

values in Table 5-2 through Table 5-5. For each problem, directed searches were run until one hundred cuts were made that did not result in an improved solution. Figure 5-10 displays convergence plots for the twenty scattered target problem. The four lines represent the best solution for the directed search over time, with each line starting at a different initial assignment. The solid blue line with circles starts from the Breadth Linear Auction solution and results in the best solution, an objective function value of 341,722. This result is a 43.47% increase over the best value from the random assignments, and is 7.63 standard deviations from the mean value. This is an increase in 0.14 standard deviations over the best solution in Table 5-2. All of the Directed Searches, starting at each of the different initial assignments, results in a solution matching or near the solution starting from the Linear Auction assignment; however, it takes a longer time to reach each of these solutions. Figure 5-11 displays the convergence of directed searches for the twenty closely grouped target problem

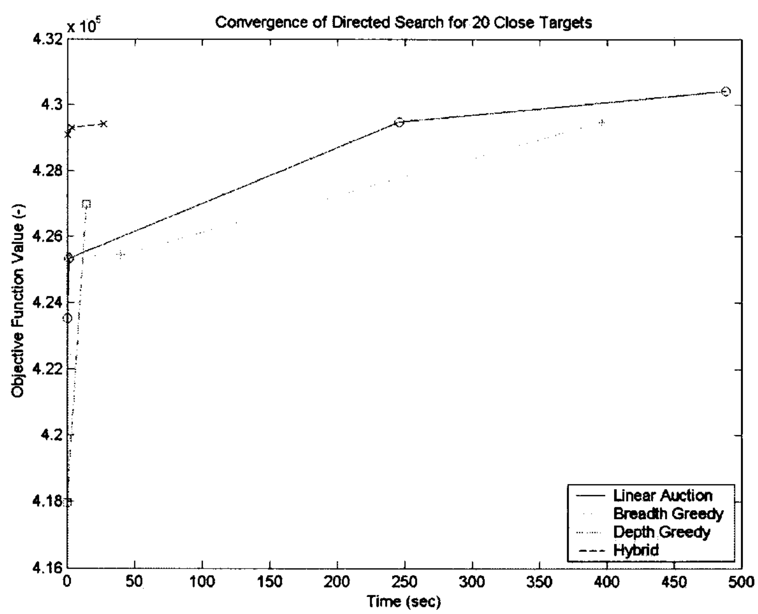


Figure 5-11. Convergence for Directed Search for 20 Close target problem from Various Starting Assignments

. Although solutions are reached for each starting assignment, different starting assignments reach different solutions more quickly. Again, the Linear Auction starting solution leads to the largest final objective function value; 430,413 in 488 seconds, which is a 43.47% increase over the maximum value from the random assignments and is 8.74 standard deviations from the mean value. This is an increase of 0.04 in the standard deviation over the best starting assignment (hybrid). Although this is the best solution, it is also the solution that takes the longest to reach. The directed search that starts from the hybrid assignment actually reaches an assignments with an objective function value of 429,419 in just over 27 seconds; an assignment whose objective function value is only 994 less than the best solution, but determined with a solution time of nearly 1/20th that of the best solution. Figure 5-12 displays the Directed Search improvement for the fifty scattered target problem. Because there are more total available solutions, this directed

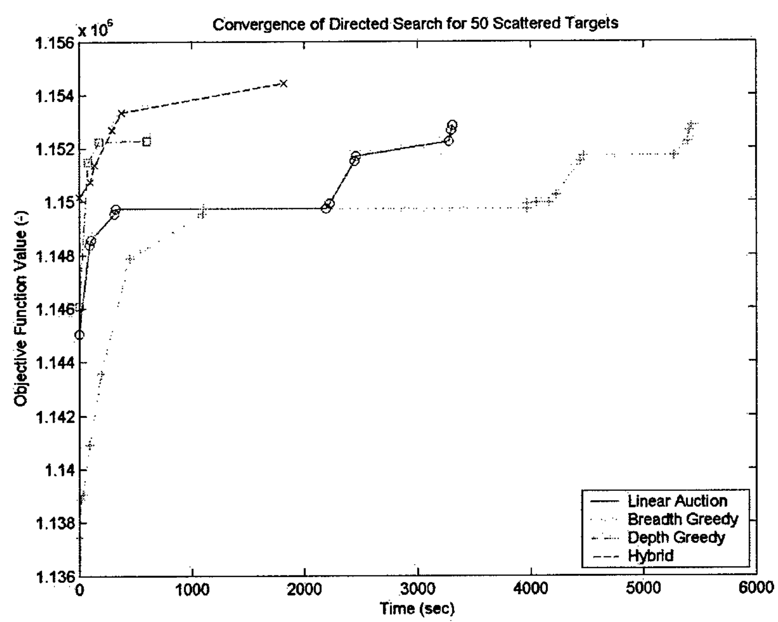


Figure 5-12. Convergence for Directed Search for 50 Scattered target problem from Various Starting Assignments

search took longer to converge to a solution, with the longest convergence taking 5,425 sec (1.5 hours). The best solution was reached by the directed search starting with the solution from the hybrid method, reaching an assignment with an objective function value of 1,154,442 in 1,825 seconds, which is an increase of 109.21% over the maximum value of the random assignments; this solution is 13.27 standard deviations from the mean of the random assignments, an improvement of 0.07 standard deviations over the best original assignment from the Hybrid method. Figure 5-13 displays the results for the

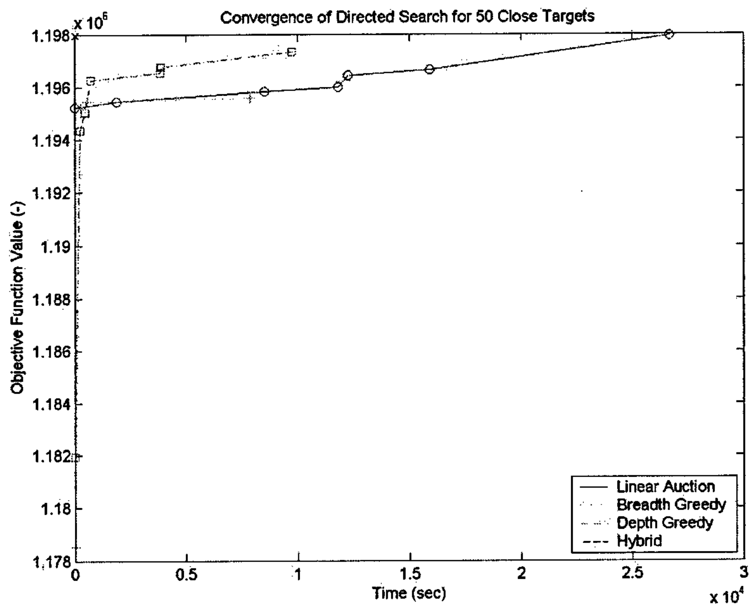


Figure 5-13. Convergence for Directed Search for 50 Close target problem from Various Starting Assignments

Directed Search problem for the fifty closely grouped target problem. Although it is difficult to see, the best solution is reached by the Directed Search method starting with the Hybrid solution. The maximum solution was 1,197,971, an improvement of only 8 over the solution from the hybrid method. This solution was reached in 38.7 seconds, therefore even though it was a relatively small increase in the solution value, it was reached rapidly.

These convergence plots show that there is not one best assignment method to use as an initial assignment, with the best assignments in the twenty target problems coming from the Linear Auction initial assignment, and the best assignment in the fifty target problems coming from the Hybrid initial assignment reached by the directed search. Although there was improvement for all problem types, the greatest improvement occurred for both of the scattered target problems, with all initial assignment directed searches showing marked improvement in Figure 5-10 and Figure 5-12. Another important observation is that the best solution from the different initial assignments, in three of the four problems, came from the Directed Search run which started from the best initial assignment. In the only problem where the best result did not come from the best starting assignment (twenty close target problem), the solution came from the second best starting assignment. Since each starting assignment takes considerably less time to calculate than the improvement from the directed search method, it makes sense to run each of the initial solution methods, and then run a directed search starting from the best initial assignment.

5.2.5 Total Solution Time using Directed Search

To compare the total solution times, using both a Breadth or Depth based method as well as a Directed Search, a table showing improvement, as well as total solution time for the four problems are given in Table 5-6 through Table 5-9. These values can be compared to the results from initial assignments in Table 5-2 through Table 5-5, showing an improvement over the results from each method. Each table displays a tradeoff between

solution and calculation speed. Although the best solutions for the twenty scattered target problem are found using the Breadth based methods, which also result in the quickest

Table 5-6. Objective Function results for 20 Scattered target problem with Directed Search

	Objective Function	Time (s)	% Imp Over Rand Max	Std Dev over Rand Mean
Breadth Based Method				
Linear Auction	341,722	95.02	43.47%	7.63
Greedy	341,722	89.45	43.47%	7.63
Depth Based Method				
Greedy	341,120	111.36	43.19%	7.61
Mixed Method				
Hybrid	341,120	108.58	43.19%	7.61

Table 5-7. Objective Function results for 20 Close target problem with Directed Search

	Objective Function	Time (s)	% Imp Over Rand Max	Std Dev over Rand Mean
Breadth Based Method				
Linear Auction	430,413	513.91	55.78%	8.74
Greedy	429,476	466.80	55.42%	8.71
Depth Based Method				
Greedy	426,987	43.69	54.48%	8.64
Mixed Method				
Hybrid	429,419	74.72	55.40%	8.71

Table 5-8. Objective Function results for 50 Scattered target problem with Directed Search

	Objective Function	Time (s)	% Imp Over Rand Max	Std Dev over Rand Mean
Breadth Based Method				
Linear Auction	1,152,861	3668.46	108.88%	13.24
Greedy	1,152,861	5692.74	108.88%	13.24
Depth Based Method				
Greedy	1,152,275	1372.03	108.75%	13.23
Mixed Method				
Hybrid	1,154,442	2880.20	109.21%	13.27

Table 5-9. Objective Function results for 50 Close target problem with Directed Search

	Objective Function	Time (s)	% Imp Over Rand Max	Std Dev over Rand Mean
Breadth Based Method				
Linear Auction	1,197,920	27164.58	118.39%	14.02
Greedy	1,195,572	8282.83	117.90%	13.98
Depth Based Method				
Greedy	1,197,321	10436.44	118.27%	14.01
Mixed Method				
Hybrid	1,197,971	1095.35	118.41%	14.02

solutions, the only other problem where the best solution is reached by the quickest solution is the fifty closely grouped problem. For both the twenty closely grouped and fifty scattered target problems, a tradeoff occurs between solution speed and solution quality. However, because all results are reached using the same Directed Search method, the main difference in solution time is the quality of the starting assignment. Therefore, the total solution time is a combination of the time to reach the starting solution, and the quality of that starting solution. In order to reach a final solution quickly, it is best to start the Directed Search with an initial assignment that can be found quickly and is close to the best final solution.

5.2.6 Results from Randomly Perturbed Weight Breadth Linear Auction

In order to estimate the success rate of the use of Randomly Perturbed weights in the Breadth Linear Auction, numerous auctions were run for each of the four problem distributions. Because each auction with randomly perturbed weights takes the same amount of time to process as a regular Breadth Linear Auction, this process can take an extremely long time. For the smaller, twenty target problems, 10,000 random runs were completed. Each of these Breadth Linear Auctions took approximately 25 seconds to complete, for a total of 250,000 seconds, or just under 70 hours. Figure 5-14 displays a histogram of results for the twenty scattered targets, with a green line representing the objective function value from the Breadth Linear Auction. As the histogram shows, very few results, 0.12% or 1 out of every 833 results, are greater than the result from the Breadth Linear Auction. Because the weights are randomly perturbed, if results follow

the distribution a better result should be reached in 833 runs (20,825 sec or 5.78 hours). Figure 5-15 is a histogram of results for the twenty closely grouped problem. This result

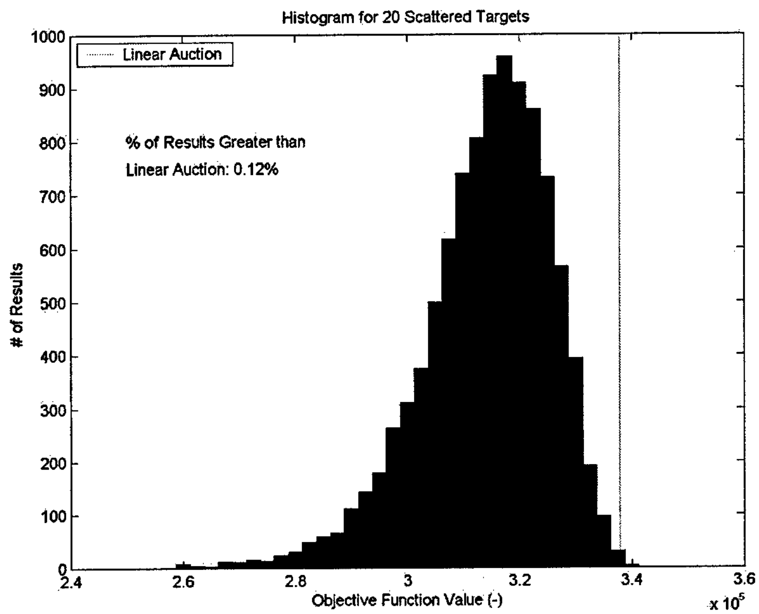


Figure 5-14. Histogram of Randomly Perturbed Weight Breadth Linear Auction for 20 Scattered targets

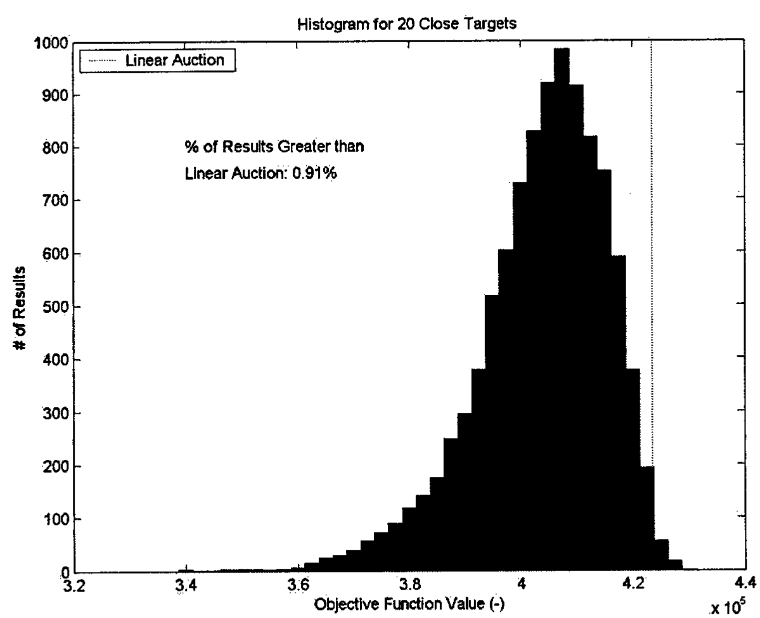


Figure 5-15. Histogram of Randomly Perturbed Weight Breadth Linear Auction for 20 Close targets

in 0.91%, or 1 out of every 110, of the results better than the initial Breadth Linear shows better performance than that for the twenty scattered problem, with more of the distribution with higher objective function values than the scattered targets. This results Auction. If results follow this distribution, a better result should be reached in 110 runs (2,750 sec or 0.78 hours). This is a much more manageable time, and could realistically be used to find better results.

For the problems with more targets (fifty), less random runs were used for the distribution due to the increase in time necessary to reach a result. Since each Linear Auction takes approximately 400 seconds to complete, 1,600 runs were used (a total of 640,000 sec or 177.77 hours). Because the data collected for both problems includes significantly less total test runs, the histograms for both problems will be less smooth and

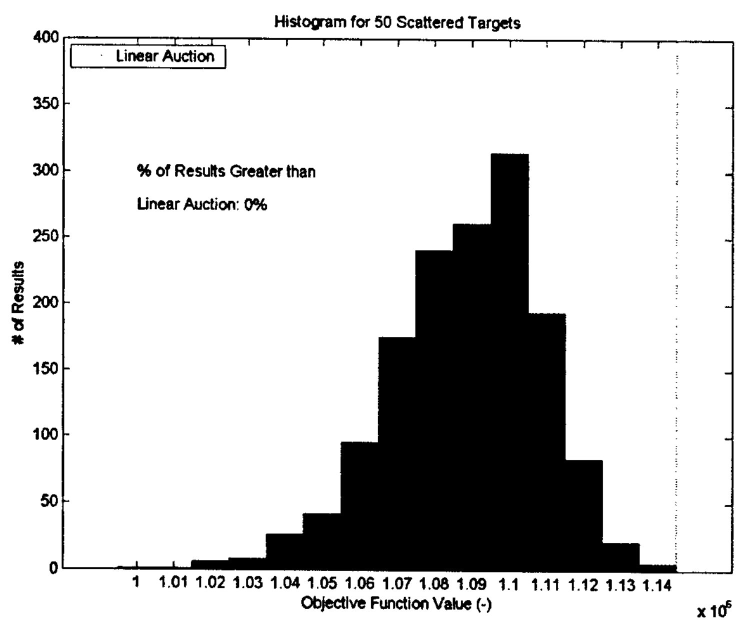


Figure 5-16. Histogram of Randomly Perturbed Weight Breadth Linear Auction for 50 Scattered targets

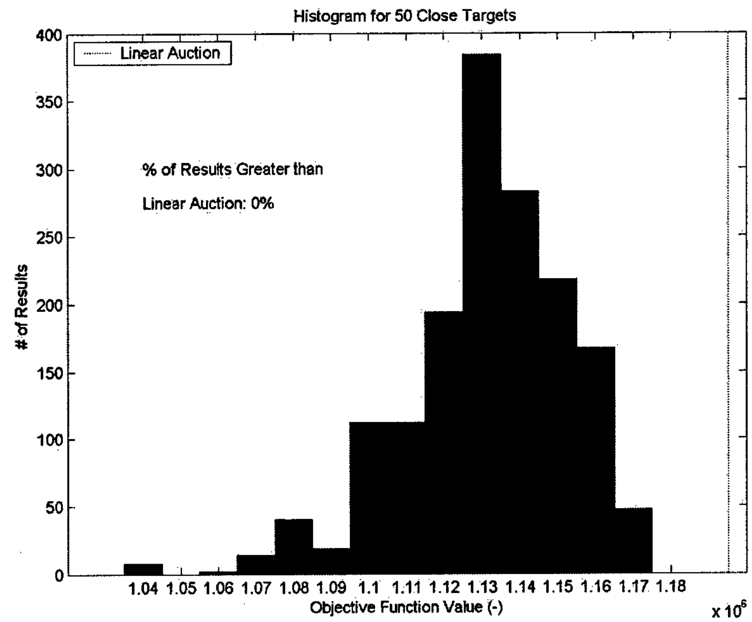


Figure 5-17. Histogram of Randomly Perturbed Weight Breadth Linear Auction for 50 Close targets

harder to use in distinguishing results. Figure 5-16 and Figure 5-17 display the histogram for both fifty target problems. Neither histogram shows any results greater than the Linear Auction result, with a large gap between the best random result and the Linear Auction result for the closely grouped targets. Therefore, it is not possible to estimate how quickly a better answer would be reached using this method. However, since it took 177.77 hours (7.4 days) to reach these results, it would not be likely that this method would result in an improved assignment in a timely manner. Therefore, although the use of Random Assignments may be a viable solution improvement method for small problems, because of the increase in total number of possible solutions and the time necessary to search each solution, it is not a viable solution improvement method for larger problems.

6 Conclusion

Target assignment involving Space Based Entry Vehicles is a complicated process. By allowing different Launch Vehicles to cross target locations with Entry Vehicles, the probability of success of these locations becomes a nonlinear function of the inclination and release time of Launch Vehicles and the assignment of each Launch Vehicle's Entry Vehicles. However, through discretization of release times and launch inclinations, the problem can be posed manageably as an assignment problem.

In order to solve the nonlinear assignment problem using linear techniques, two basic methods of problem decomposition, breadth and depth based methods, are used. One unique implementation of the breadth based method involves the use of numerical Linear Auctions to make assignments. This method effectively optimizes assignments for each cycle of the breadth based method, solving for the optimal solution to linear assignment problems. A hybrid method is also used, which takes advantage of the strengths of both methods, at the cost of a large increase in computational effort. Although both the breadth and depth methods have strengths and weaknesses, neither method guarantees an optimal solution. Similarly, both methods do not include the ability to improve on the assignment solution.

In order to overcome some of the weaknesses of both methods, a Directed Search is used to improve on assignment solutions. Using the Breadth Linear Auction, the Directed Search method determines assignments that do not increase the overall utility as much as other assignments. Because of their poor performance, these assignments are eliminated,

and the process iterates on solutions, attempting to find better assignments. Although this method does not guarantee increases in the objective function during each cycle, by completing numerous cycles better solutions will be determined. An alternative method to improve the solution includes the uses of randomly perturbed weights within the Breadth Linear Auction solution method. By perturbing the weights randomly, the search space near the original assignment solution can be searched by running numerous Breadth Linear Auctions. Because of the randomness of this process, this method can take long periods of time to reach better solutions. Therefore, although it can be used for small problems, it is not suggested for use with larger problems.

To examine the strengths and weaknesses of the breadth and depth based methods, two problem variations were developed which displayed the strengths and weaknesses of both methods. By running each problem with the same target locations but varying target weights, it's possible to find situations where both methods have difficulties making assignments due to the lack of an ability to predict the effect each assignment will have on the ability to make effective future assignments.

In order to compare the different solution methods' ability to make assignments, four problems with varying numbers of targets and target grouping were developed. For each problem, the Breadth Linear Auction outperformed the Breadth Greedy method, however with relatively small increases in utility. The Hybrid method resulted in the best solution to three of the four problems, however again with relatively small improvements in results, and with much greater computational effort in reaching solutions when compared

to the other methods. When improving on the solutions using the Directed Search, solution improvement over time was compared for different starting solutions. The improvement in solutions was best for Directed Searches which started with a Breadth Linear Auction solution for problems with small numbers of targets, and best when started with a Hybrid solution for problems with larger numbers of targets. In order to estimate the effectiveness of using Random Weights in the Breadth Linear Auction, numerous solutions were reached and a histogram of solutions was plotted. For problems with smaller numbers of targets, improvements in solutions are estimated to occur in 1% of random assignments; however, no improvements in solutions ever occurred for larger problems. Therefore, the use of random weights may be a viable solution improvement method for small problems, but impractical for larger problems.

Future research in target assignment for space based entry vehicles should include a differentiation between Launch Vehicles based on estimated time to target. Because all Launch Vehicles will not be launched simultaneously, either due to limitations on the number of launch sites or the necessity to launch in waves, it is important to consider time to target. This is especially true when targets are mobile or time sensitive. Incorporating a penalty for destroying a target at a later time will cause the objective function to become more complicated, and may require changes to the different assignment methods. Another area of future research should include integration into a large, campaign planning framework that includes other weapons, such and land, sea, and air based assets. Figure 6-1 displays the

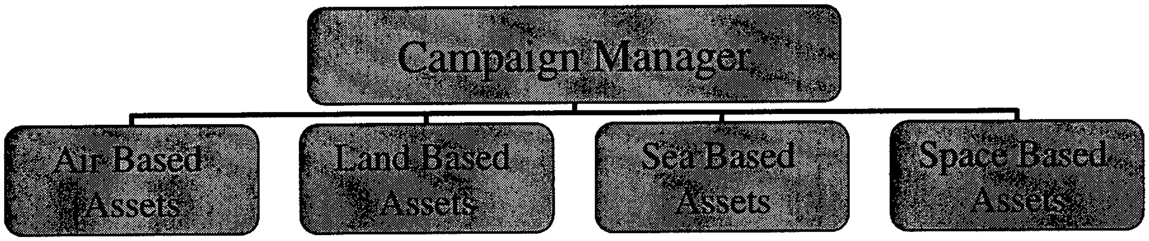


Figure 6-1. Diagram of Campaign Manager Integration

integration of the Space Based assignment process into the overall campaign framework. Such integration would require an iterative process where the campaign manager would give certain targets to the Space Based Assets target assignment process, the process would determine the best assignment of targets to LVs, and then return the P_S of destroying these targets. If this P_S was not acceptable by the manager, a different set of targets would be given to the Space Based EV assignment process, resulting in a new P_S. This process would continue until the Campaign manager decides which locations to target with the Space Based Assets. In such a system, rapid solution times would be very important.

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